T5 C22 Parametric Functions Arc Length

Consider the graph of $(f(t),g(t))$ from $t = a$ to b.

What will be its arc length?

Divide $[a,b]$ into many very small subintervals

$a=t_0$, t_1 , t_2 , … t_i , $t_i + h$, … $t_n =b$ where $h =(b-a)/n$

The length of the segment of the curve from t_i to $t_i + h$,

By Pythagorus will be

$[(g(t_i + h) - (g(t_i))^2 + (f(t_i + h) - (f(t_i))^2]^{1/2}$

$=h[\{(g(t_i + h) - (g(t_i))/h\}^2 + \{(f(t_i + h) - (f(t_i))/h\}^2]^{1/2}$

$= [(g'(t))^2 +(f'(t))^2]^{1/2}\,dt$ when h is an infinitesimal dt.

Add these all up with integration and we get

Arc Length = Integral $[(g'(t))^2 +(f'(t))^2]^{1/2}dt$ from a to b.

You might want to compare this to the argument we gave for arc length of $y = f(x)$ from $x = a$ to b. Very similar.

Parametric function representation of a curve is just a more general way to do it.

After all, if we let $x = t$, then $(t, f(t))$ from a to b, is just the parametric function representation of this same function where $x = t$ and $y = f(t)$.

And Integral $[1 + (f'(t))^2]^{1/2}dt$ from $t = a$ to b

is the arc length of the curve.

Now, we observed that WA calculated the arc length of the various examples we did before.

WA 1 Parametric Plot (t^2,t) from t = -2 to 2

But, we could have just asked for the arc length directly

WA 2 Arc Length (t^2,t) from t = -2 to 2

Note: To evaluate this it was necessary to find the anti-derivative. Classically, this was often a challenge, and sometimes simply impossible.

WA 3 Integrate (1 + 4t^2)^.5

Notice the anti-derivative. You can verify it by differentiation, but how would you find it?

It is always good to verify a formula you have derived with a known example. What is more familiar than the circle?

WA 4 Arc Length (cos(t), sin(t)) from t=0 to 2Pi

Note: The integrand, $1 = [(-\sin(t))^2 + (\cos(t))^2]^{1/2}$

Sometimes, life is easy.

Sometimes not.

Ellipses are critical in physics since they describe the orbits of the planets about the sun, and moons about the planets.

Newton proved this in the late 1600's after Kepler derived the equations from astronomical observations data.

So, Simply try to find the arc length, circumference, of a semi-ellipse like our ancestors had to.

WA 5 Arc Length (5cos(t), 7sin(t)) from t = 0 to Pi

Well, the answer is 18.98.

But, how did WA find this?

WA 6 Integrate (37 + 12cos(2t))^.5

This is an example of what is called an elliptic integral.

In integral calculus, elliptic integrals originally arose in connection with the problem of giving the arc length of an ellipse. They were first studied by Giulio Fagnano and Leonhard Euler. Modern mathematics defines an "elliptic integral" as any function *f* which can be expressed in the form

$$f(x) = \int_c^x R\left(t, \sqrt{P(t)}\right)\, dt,$$

where *R* is a rational function of its two arguments, *P* is a polynomial of degree 3 or 4 with no repeated roots, and *c* is a constant.

In general, integrals in this form cannot be expressed in terms of elementary functions. Exceptions to this general rule are when *P* has repeated roots, or when *R*(*x*,*y*) contains no odd powers of *y*. However, with the appropriate reduction formula, every elliptic integral can be brought into a form that involves integrals over rational functions and the three Legendre canonical forms (i.e. the elliptic integrals of the first, second and third kind).

In general, this is an example of a function that does not have an anti-derivative consisting of well known simple functions covered in calculus.

Unfortunately, there are many such functions. In fact, when setting up a math model in a STEM subject, this is often the situation.

Historically, STEM pros would simplify their models so they could solve them with the standard calculus tools. This, of course, led to models that were not very accurate.

Fortunately, today with tools like WA and Mathematica one can easily evaluate such integrals numerically and, thus, not worry about restricting one's self to the classical tools of calculus.

Our ancestors would have given an awful lot to have such a tool. They spent immense amounts of time manually calculating such things and compiling extensive tables of what are called special functions.

Of course, they weren't distracted or entertained by such things as TV, video games, Internet, smart phones, or even good music. So, I suppose they found joy in these calculations that most of us today would find stultifying and boring and a waste of time.

You may want to go back and look at the examples of the previous lesson and see how the arc lengths are calculated.

Furthermore, our ancestors spent a lot of time finding antiderivatives of functions so they could apply the FTC to evaluate definite integrals.

Indeed, many calculus courses still spend a lot of time on these "techniques of integration". But, with tools like WA you probably will never use them, just like you probably won't use a slide rule, even though it is much superior to pencil and paper calculations.

Personally, I would not go to a school that made me learn obsolete anachronistic tools.

T5 C22 Parametric Functions – Arc Length Exercises

Q1. Given the function $x = y^4$, find the arc length using t as an independent variable, from $-2<t<2$.

Q2. Given the function $x = y^4 + 3y^3 - 10y^2$, find the arc length using t as an independent variable, from $-6<t<6$.

Q3. Given the function $x = e^y - y^2$, find the arc length using t as an independent variable, from $-8<t<8$.

Q4. Given the function for a circle $x^2 + y^2 = 25$, find the arc length using t as an independent variable, from $t = 0$ to 2π.

Q5. Given the function for a circle $x^2 + y^2 = 9$, find the arc length using t as an independent variable, from $t = 0$ to 2π.

Q6. Given the function for an ellipse $x^2/9 + y^2/4 = 1$, find the arc length using t as an independent variable, from $t = 0$ to 2π.

Q7. Given the function for an ellipse $x^2/25 + y^2 = 1$, find the arc length using t as an independent variable, from $t = 0$ to 2π.

Q8. Given the following functions, find the arc length: $f(t) = t^3 - 2\sin(t)$, $g(t) = 3t + 2\sin(3t)$ from $-10<t<10$.

Q9. Given the following functions, find the arc length: $f(t) = t^2 - \sin(t)$, $g(t) = t + 3\sin(3t)$ from $-5<t<5$.

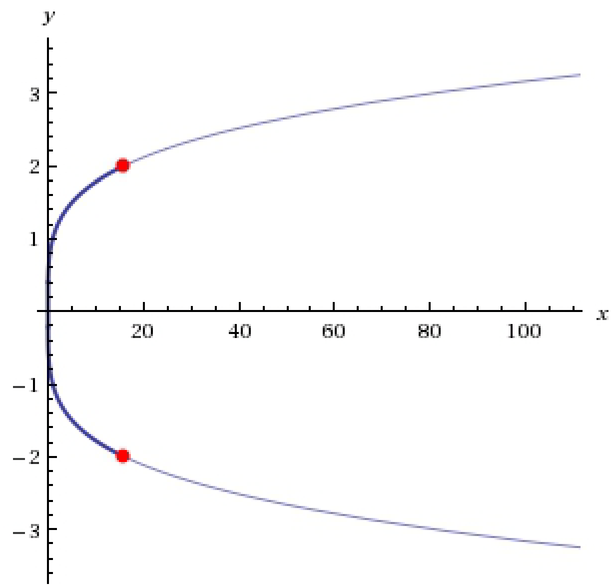
Q10. Given the following functions, find the arc length: $f(t) = 3t^3 - 2\sin(t)$, $g(t) = 3t + 4\sin(5t)$ from $-4<t<7$.

A1. WA arc length (t^4, t) from -2<t<2

Result:

$$\int_{-2}^{2} \sqrt{1 + 16\,t^6}\; dt \approx 33.2937\ldots$$

Parametric plot:



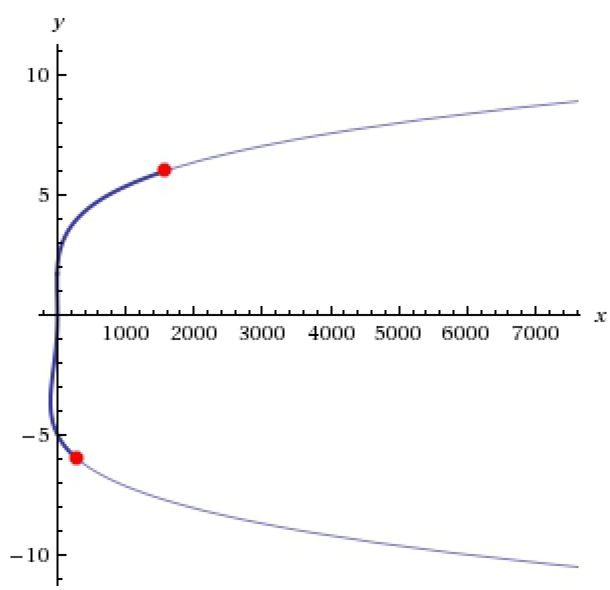
.

A2. WA arc length (t^4 + 3t^3 – 10t^2, t) from -6<t<6

Result:

$$\int_{-6}^{6} \sqrt{1 + t^2\left(-20 + 9\,t + 4\,t^2\right)^2}\; dt \approx 2090.809\ldots$$

Parametric plot:

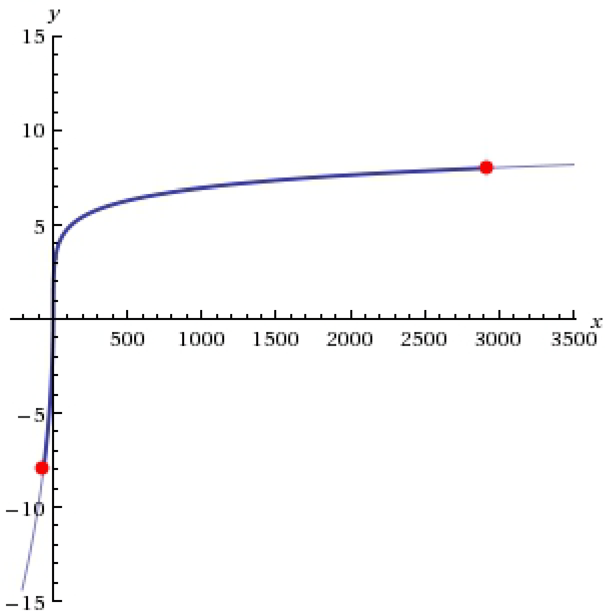


A3. WA arc length (e^t - t^2, t) from -8<t<8

Result:

$$\int_{-8}^{8} \sqrt{1+\left(e^t - 2\,t\right)^2}\; dt \approx 2982.686\ldots$$

Parametric plot:

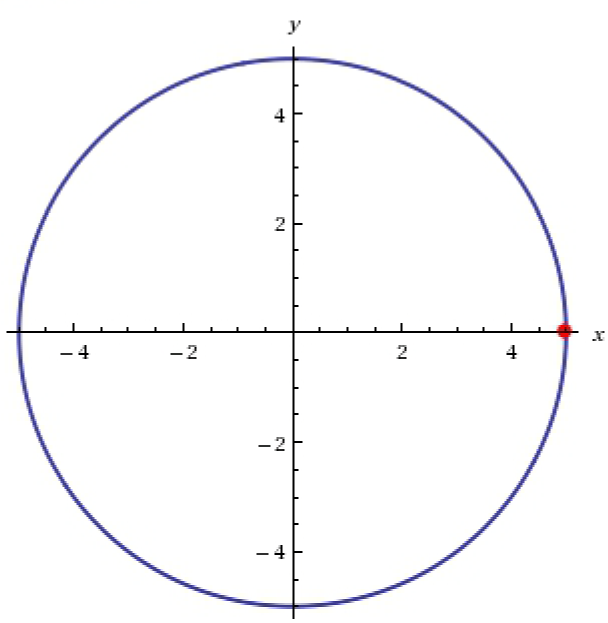


A4. WA arc length (5cos(t),5sin(t)) from t = 0 to 2pi

Result:

$$\int_0^{2\pi} 5\, dt = 10\,\pi \approx 31.4159$$

Parametric plot:

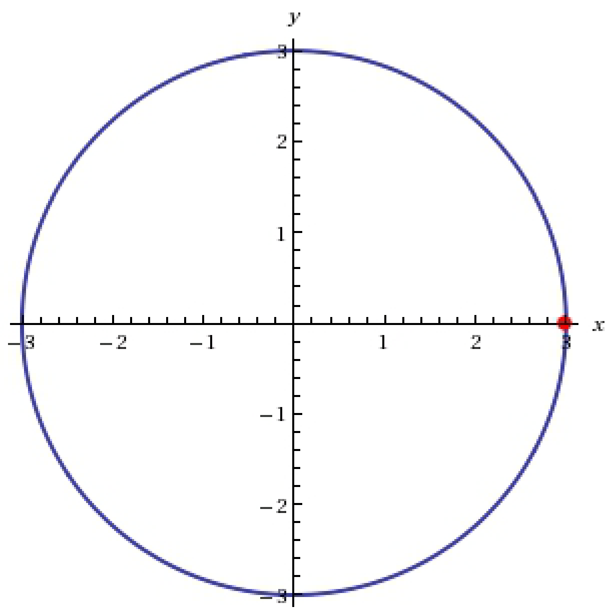


A5. WA arc length (9cos(t),9sin(t)) from t = 0 to 2pi

Result:

$$\int_0^{2\pi} 3\,dt = 6\pi \approx 18.8496$$

Parametric plot:



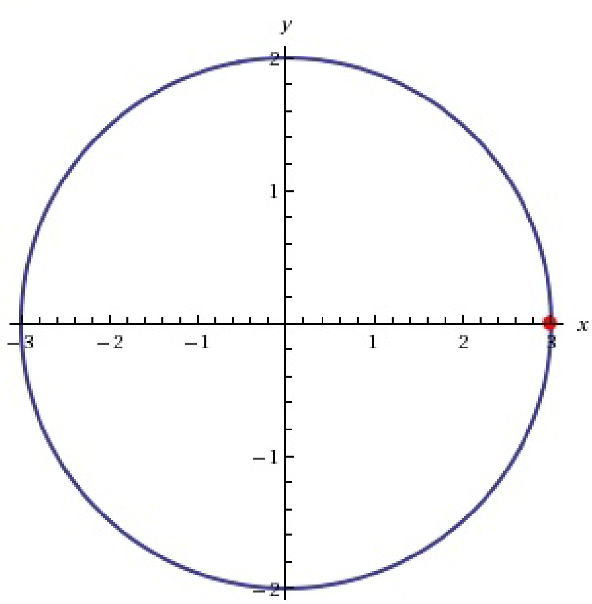
A6. WA arc length (3cos(t),2sin(t)) from t =0 to 2Pi

Result:

$$\int_0^{2\pi} \sqrt{4\cos^2(t) + 9\sin^2(t)}\,dt = 8\,E\left(-\frac{5}{4}\right) \approx 15.8654$$

$E(m)$ is the complete elliptic integral of the second kind with parameter $m = k^2$

Parametric plot:



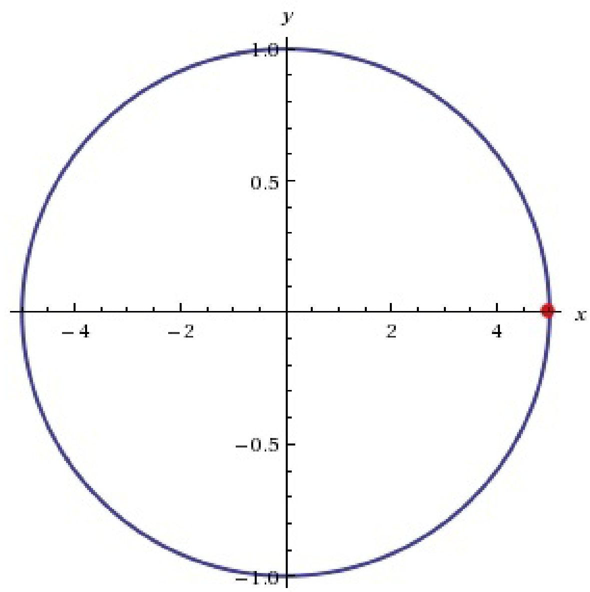
A7. WA arc length (5cos(t),sin(t)) from t =0 to 2Pi

Result:

$$\int_0^{2\pi} \sqrt{13 - 12\cos(2t)}\, dt = 4E(-24) \approx 21.01$$

$E(m)$ is the complete elliptic integral of the second kind with parameter $m = k^2$

Parametric plot:

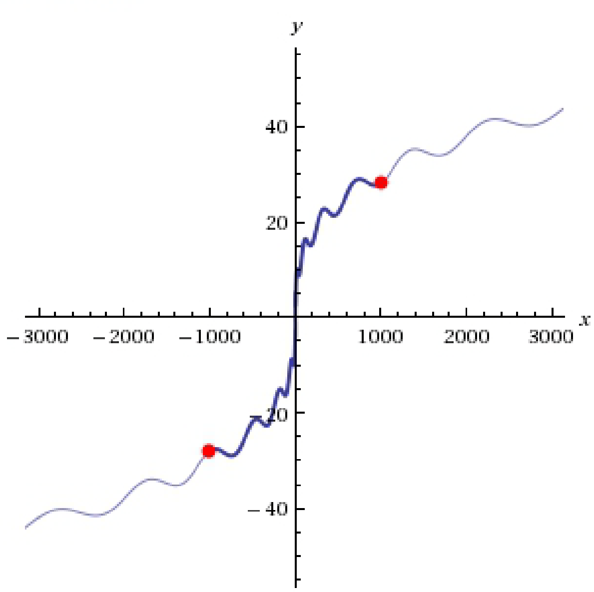


A8. WA arc length (t^3 - 2sin(t), 3t + 2sin(3t)) from -10<t<10

Result:

$$\int_{-10}^{10} \sqrt{\left(3t^2 - 2\cos(t)\right)^2 + (3 + 6\cos(3t))^2}\, dt \approx 2018.8\ldots$$

Parametric plot:

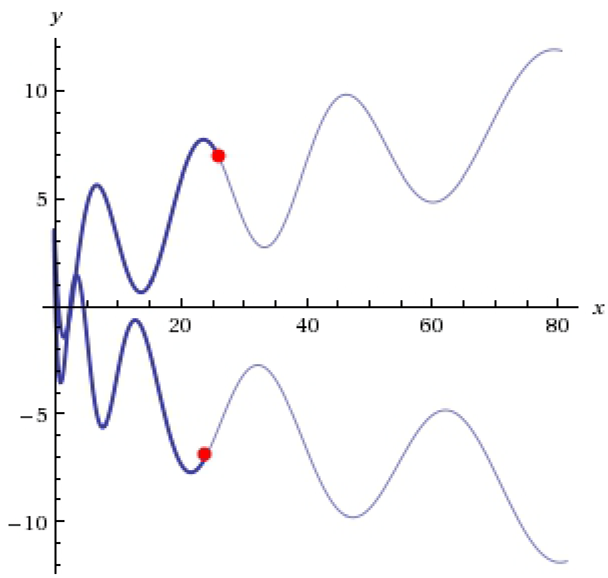


A9. WA arc length (t^2 - sin(t), t + 3sin(3t)) from -5<t<5

Result:

$$\int_{-5}^{5} \sqrt{(-2\,t + \cos(t))^2 + (1 + 9\cos(3\,t))^2}\; dt \approx 81.8744\ldots$$

Parametric plot:

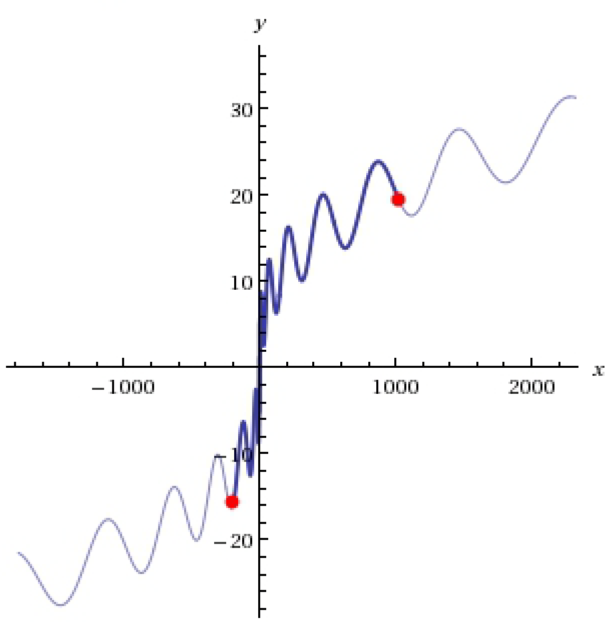


A10. WA arc length (3t^3 - 2sin(t), 3t + 4sin(5t)) from -4<t<7

Result:

$$\int_{-4}^{7} \sqrt{\left(9\,t^2 - 2\cos(t)\right)^2 + (3 + 20\cos(5\,t))^2}\; dt \approx 1261.7\ldots$$

Parametric plot:



T5 C23 Parametric Functions Tangent Line

Let G be the graph defined by $(f(t),g(t))$ from $t = a$ to b.

Consider that this graph is in the x-y plane coordinate system.

For a specific value of t, say $t = c$, what would be the equation of the tangent line to the graph at $(f(c),g(c))$?

If you are a potential math major I recommend you try to figure this out on your own.

Obviously, this tangent line passes through $(f(c), g(c))$ and has a slope, m, which of course depends on c. The problem is to figure out the slope, m.

Then the equation the tangent line will be

$y – g(c) = m(x – f(c))$.

So what is m?

Well, let's consider an approximation of the slope m by adding an infinitesimal h to c.

Now, what would be the slope of the line segment from

$(f(c), g(c))$ to $(f(c + h), g(c + h))$?

Clearly $m \approx [g(c + h) - (g(c)] / [f(c + h) - (f(c)]$

$= \{[g(c + h) - (g(c)] / h\} / \{[f(c + h) - (f(c)] / h\}$

So, letting h be an infinitesimal we obtain

$m = g'(c) / f'(c)$ So,

$$y – g(c) = [g'(c) / f'(c)](x – f(c)]$$

is the equation of the tangent line.

Let's start with a couple of examples we already know the answer to and use WA even though we could do it in our heads.

Example 1. Find the tangent line to the parabola (t, t^2) at (2,4) Of course, we know the equation will be

y – 4 = 4(x – 2) or y = 4x - 4 from what we learned earlier.

WA1 tangent line to y = x^2 at x = 2

Now let's do it parametrically

WA2 (Derivative t^2)/(Derivative t) , t=2

So we see m = 4 and the equation follows.

Or, we can obtain the equation of the tangent line directly

WA3 y - t^2 =((Derivative t^2)/(Derivative t))(x-t) , t=2

Real Solution: y = 4x - 4

Example 2. Find the tangent line to the circle

(5cos(t), 5sin(t)) at (5cos(Pi/3), 5sin(Pi/3))

WA4 Parametric Plot (5cos(t),5sin(t)) from t = 0 to 2Pi

WA5 y - 5sin(t) = ((Derivative 5sin(t))/(Derivative 5cos(t)))(x-5cos(t)) , t=Pi/3

Or, we can enter an approximation number Pi/3 = 1.047

WA6 y - 5sin(t) = ((Derivative 5sin(t))/(Derivative 5cos(t)))(x-5cos(t)) , t=1.047

A longer way

WA7 (5cos(Pi/3), 5sin(Pi/3))

Ans. $(5/2, 5X3^{1/2}/2) = (2.50, 4.33)$

WA8 (Derivative 5sin(t))/(Derivative 5cos(t)) , t= Pi/3

$m = -3^{-1/2} = -.577$ so $y - 4.33 = -.577(x - 2.50)$

0r $y = -.577x + 5.77$

Observe: When $g'(c) = 0 \neq f'(c)$ we have a horizontal tangent, and when $g'(c) \neq 0 = f'(c)$ we have a vertical one.

WA9 (Derivative 5sin(t))/(Derivative 5cos(t)) , t= Pi/2

Let's find all of the horizontal tangents

WA10 Solve Derivative 5sin(t) = 0

Pi/2, 3Pi/2, etc. (n – ½)Pi for any integer n.

How about the vertical tangents?

WA11 Solve Derivative 5cos(t) = 0

nPi for all integers, of course. We knew this.

Let's just verify it.

WA12 (Derivative 5sin(t))/(Derivative 5cos(t)) , t= Pi

Of course, $m = \infty$

Example 3 Find the tangent line to the graph of the ellipse (5cos(t), 7sin(t)) at (5cos(5Pi/4), 7sin(5Pi/4).

WA13 Parametric Plot (5cos(t), 7sin(t)) from t = 0 to 2Pi

WA14 (5cos(Pi/4), 7sin(Pi/4))

OK $(5/2^{1/2}, 7/2^{1/2}) \approx (3.54, 4.95)$

WA15 (Derivative 7sin(t))/(Derivative 5cos(t)) , t= Pi/4

m = -7/5 = -1.4 so y – 4.95 = -1.4(x – 3.54) 0r

y = -1.4x + 9.9 is tangent line.

WA16 y - 7sin(t) = ((Derivative 7sin(t))/(Derivative 5cos(t)))(x-5cos(t)) , t=.785

You can find the horizontal and vertical tangents.

Example 4. Find the tangent line to the graph of the ellipse (5cos(t), 7sin(t)) at t = 2.25

WA13 Parametric Plot (5cos(t), 7sin(t)) from t = 0 to 2Pi

WA 17 (5cos(2.25), 7sin(2.25))

WA18 y - 7sin(t) = ((Derivative 7sin(t))/(Derivative 5cos(t)))(x-5cos(t)) , t=2.25

Summary:

I. Parametric Plot the graph

II. Find the point on the graph

III. Find the Slope at that point

IV. Apply analytical geometry for straight line.

Or in place of II, III, and IV

Simply use the WA instruction

y – g(t) = ((derivative g(t))/(derivative f(t)))(x – f(t)) at t=a

With WA this all is very easy.

For many such problems in the old days, it was difficult.

Not as bad as integration problems, but still time consuming and messy.

If you do ten of these types of problems it will become very easy.

T5 C23 Parametric Functions – Tangent Line Exercises

Q1. Find the slope, then find the tangent line to (t, t^3) at (2,8).

Q2. Find the slope, then find the tangent line to (t, t^4) at (2,16).

Q3. Find the slope, then find the tangent line to the circle (3cos(t), 3sin(t)) at (3cos(Pi/5), 3sin(Pi/5)) using an approximate number for Pi/5.

Q4. Find the tangent line to the circle (3cos(t), 3sin(t)) at (3cos(2Pi/3), 3sin(2Pi/3)) using an approximate number for 2Pi/3.

Q5. Find the tangent line to the circle (4cos(t), 4sin(t)) at (4cos(4Pi/5), 4sin(4Pi/5)) using an approximate number for 4Pi/5.

Q6. Find the tangent line to the circle (4cos(t), 4sin(t)) at (4cos(2Pi/3), 4sin(2Pi/3)) using an approximate number for 2Pi/3.

Q7. Find the slope, then find the tangent line to the graph of the ellipse (2cos(t), 5sin(t)) at (2cos(5Pi/3), 5sin(5Pi/3), using an approximate number for 5Pi/3.

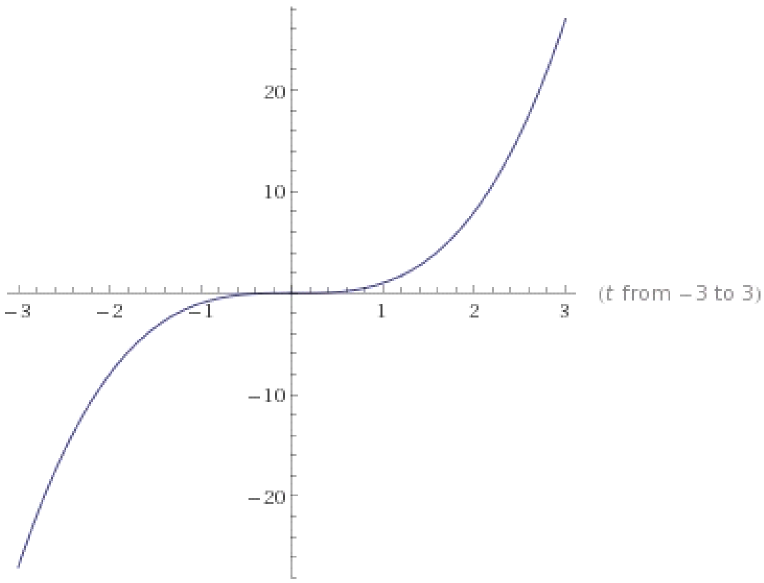
Q8. Find the slope, then find the tangent line to the graph of the ellipse (5cos(t), 2sin(t)) at (5cos(5Pi/3), 2sin(5Pi/3), using an approximate number for 5Pi/3.

Q9. Find the tangent line to the graph of the ellipse (3cos(t), 8sin(t)) at (3cos(Pi/4), 8sin(Pi/4), using an approximate number for Pi/4.

Q10. Find the tangent line to the graph of the ellipse (3cos(t), 8sin(t)) at (3cos(Pi/3), 8sin(Pi/3), using an approximate number for Pi/3.

A1. WA parametric plot (t, t^3) from t=-3 to 3

Parametric plot:



WA (Derivative t^3)/(Derivative t) , t=2

Input interpretation:

$$\left\{\frac{\frac{\partial t^3}{\partial t}}{t'(t)},\ t = 2\right\}$$

Result:

$$\{3\,t^2,\ t = 2\}$$

Substitution:

$$3\,t^2 = 12$$

WA y - t^3 =((Derivative t^3)/(Derivative t))(x-t) , t=2

Real solution:

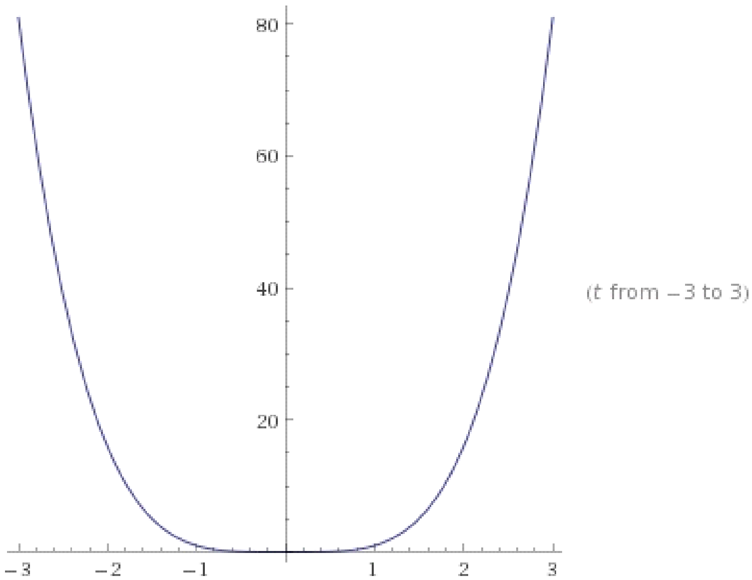
$$t = 2,\quad y = 12\,x - 16$$

Solution:

$$t = 2,\quad y = 4\,(3\,x - 4)$$

A2. WA parametric plot (t, t^4) from t=-3 to 3

Parametric plot:



WA (Derivative t^3)/(Derivative t) , t=2

Result:

$\{4t^3, t = 2\}$

Substitution:

$4t^3 = 32$

WA y - t^4 =((Derivative t^4)/(Derivative t))(x-t) , t=2

Real solution:

$t = 2, \quad y = 32x - 48$

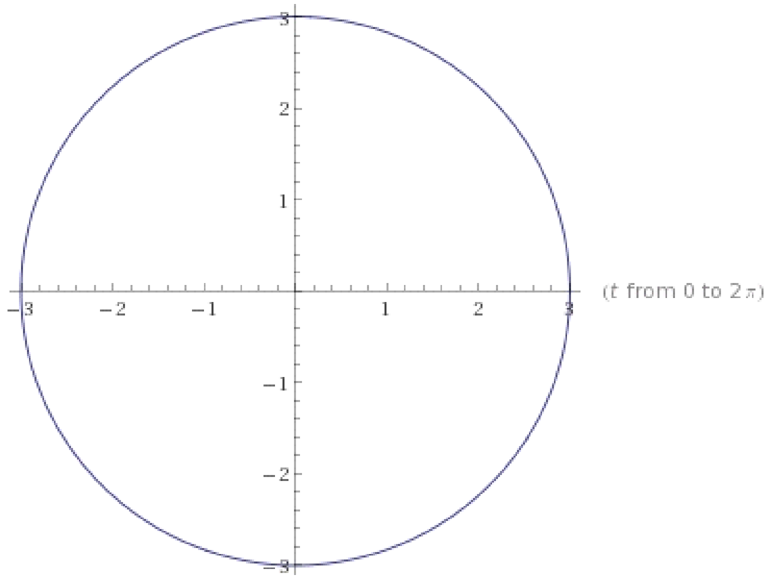
Solution:

$t = 2, \quad y = 16(2x - 3)$

A3. Pi/5 = 0.628

WA Parametric Plot (3cos(t), 3sin(t)) from t = 0 to 2Pi

Parametric plot:



WA (Derivative 3sin(t))/(Derivative 3cos(t)) , t=0.648

Result:

$\{-\cot(t), t = 0.648\}$

Substitution:

$-\cot(t) \approx -1.32091$

WA y - 3sin(t) = ((Derivative 3sin(t))/(Derivative 3cos(t)))(x-3cos(t)), t=0.648

Real solution:

$t \approx 0.648, \quad y \approx 4.97024 - 1.32091\,x$

Solution:

$t \approx 0.648, \quad y \approx 4.97024 - 1.32091\,x$

A4. 2Pi/3 = 2.094

WA Parametric plot (3cos(t), 3sin(t)) from t=0 to 2Pi

(See previous problem for plot. It is the same.)

WA y - 3sin(t) = ((Derivative 3sin(t))/(Derivative 3cos(t)))(x-3cos(t)), t=2.094

Real solution:

$t \approx 2.094, \quad y \approx 0.576824\, x + 3.46331$

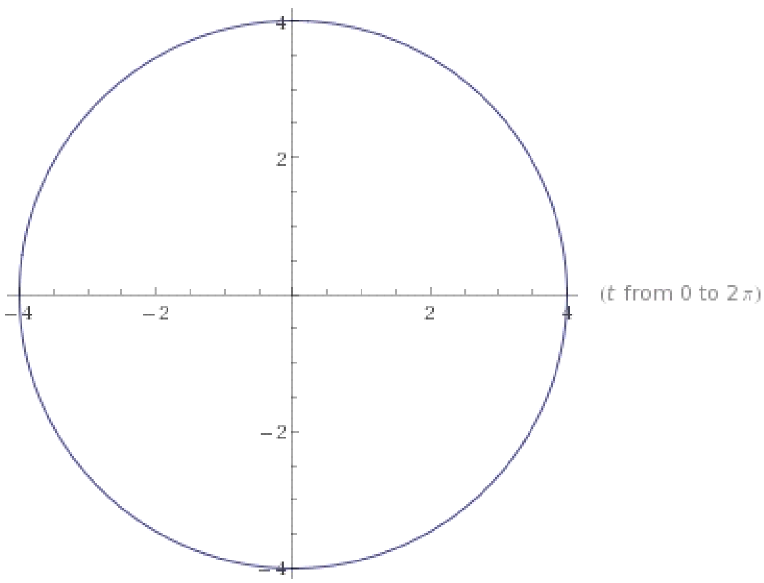
Solution:

$t \approx 2.094, \quad y \approx 0.576824\, x + 3.46331$

A5. 4Pi/5 = 2.513

WA Parametric plot (4cos(t), 4sin(t)) from t=0 to 2Pi

Parametric plot:



WA y - 4sin(t) = ((Derivative 4sin(t))/(Derivative 4cos(t)))(x-4cos(t)), t=2.513

Real solution:

$t \approx 2.513, \quad y \approx 1.37559\, x + 6.80264$

Solution:

$t \approx 2.513, \quad y \approx 1.37559\, x + 6.80264$

A6. 2Pi/3 = 2.094

WA Parametric plot (4cos(t), 4sin(t)) from t=0 to 2Pi

(See previous problem for plot. It is the same.)

WA y - 4sin(t) = ((Derivative 4sin(t))/(Derivative 4cos(t)))(x-4cos(t)), t=2.094

Real solution:

$t \approx 2.094, \quad y \approx 0.576824\,x + 4.61775$

Solution:

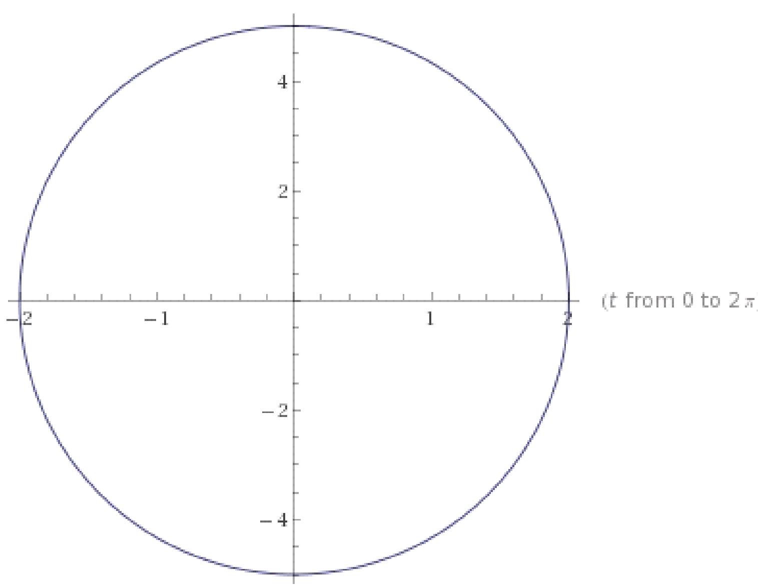
$t \approx 2.094, \quad y \approx 0.576824\,x + 4.61775$

Look back to Q4. Note that though the radius of the circle changes, the slope at the same position (2Pi/3) remains the same. The y-intercept of the tangent line, however, does change.

A7. 5Pi/3 = 5.236

WA Parametric Plot (2cos(t), 5sin(t)) from t = 0 to 2Pi

Parametric plot:



WA (Derivative 5sin(t))/(Derivative 2cos(t)) , t= 5.236

Result:

$$\left\{-\frac{5\cot(t)}{2}, t = 5.236\right\}$$

Substitution:

$$-\frac{5\cot(t)}{2} \approx 1.44342$$

WA y - 5sin(t) = ((Derivative 5sin(t))/(Derivative 2cos(t)))(x-2cos(t)) , t=5.236

Real solution:

$t \approx 5.236, \quad y \approx 1.44342\, x - 5.77354$

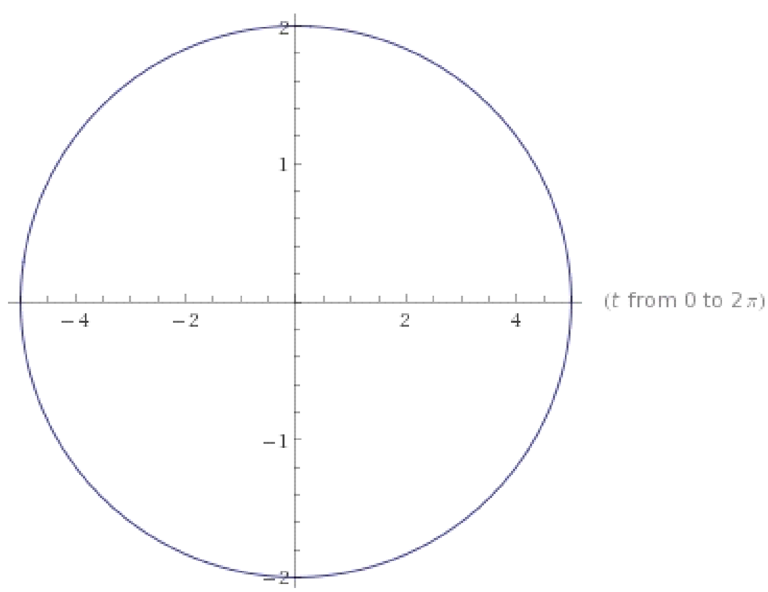
Solution:

$t \approx 5.236, \quad y \approx 1.44342\, x - 5.77354$

A8. 5Pi/3 = 5.236

WA Parametric Plot (5cos(t), 2sin(t)) from t = 0 to 2Pi

Parametric plot:



WA (Derivative 2sin(t))/(Derivative 5cos(t)) , t= 5.236

Result:

$$\left\{-\frac{2\cot(t)}{5},\, t = 5.236\right\}$$

Substitution:

$$-\frac{2\cot(t)}{5} \approx 0.230947$$

WA y - 2sin(t) = ((Derivative 2sin(t))/(Derivative 5cos(t)))(x-5cos(t)) , t=5.236

Real solution:

$t \approx 5.236\,,\quad y \approx 0.230947\,x - 2.30942$

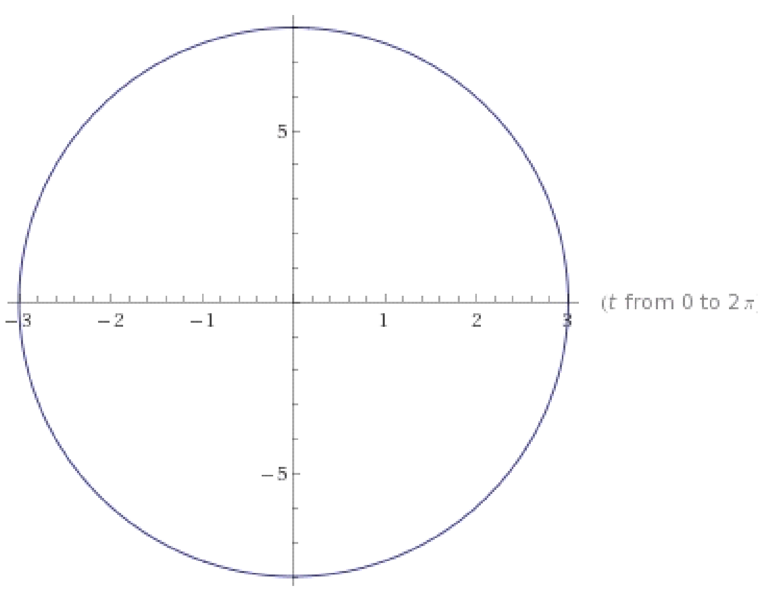
Solution:

$t \approx 5.236\,,\quad y \approx 0.230947\,x - 2.30942$

A9. Pi/4 = 0.785

WA Parametric Plot (3cos(t), 8sin(t)) from t = 0 to 2Pi

Parametric plot:



WA y - 8sin(t) = ((Derivative 8sin(t))/(Derivative 3cos(t)))(x-3cos(t)) , t=0.785

Real solution:

$t \approx 0.785, \quad y \approx 11.3182 - 2.66879\,x$

Solution:

$t \approx 0.785, \quad y \approx 11.3182 - 2.66879\,x$

A10. Pi/3 = 1.047

WA Parametric Plot (3cos(t), 8sin(t)) from t = 0 to 2Pi

(See previous problem for plot. It is the same.)

WA y - 8sin(t) = ((Derivative 8sin(t))/(Derivative 3cos(t)))(x-3cos(t)) , t=1.047

Real solution:

$t \approx 1.047, \quad y \approx 9.23866 - 1.5403\,x$

Solution:

$t \approx 1.047, \quad y \approx 9.23866 - 1.5403\,x$

T5 C24 Parametric Functions   Area

Suppose we have a function, $y = F(x)$ from a to b that is represented parametrically by $(f(t),g(t)$ from c to d.

That is, $(a,F(a)) = (f(c),g(c))$ and $(b,F(b)) = (f(d),g(d))$

And $(x,F(x)) = (f(t),g(t))$ for some t.

We know the area under the graph of $y = F(x)$ from a to b is Integral $F(x)dx$ from a to b

How could this area be calculated using $(f(t),g(t))$?

Answer; Integral $f'(t)g(t)dt$ from c to d

Let's see why

Select a value for t in the interval (c,d) and let h be an infinitesimal, later we will call it dt.

What is the area of the rectangle determined by $(f(t),g(t))$ and $(f(t+h),g(t+h))$?

See Diagram

The four corners of the rectangle are:

$(f(t),0)$ $(f(t+h),0)$ $(f(t),g(t))$ $(f(t+h),g(t+h))$

Its area is $\approx$

$[f(t+h) - f(t)]xg(t) = \{[f(t+h) - f(t)]/h\}xg(t)xh$

$=f'(t)g(t)dt$

So the total area is just the sum of all these infinitesimal rectangles, or Integral $f'(t)g(t)dt$ from c to d

I might point out that this type of infinitesimal argument was just what our ancestors used when the calculus was being invented and developed.

And, today these arguments can be made completely rigorous using the Hyperreals developed in the 1960's, sometimes called nonstandard analysis.

Example 1. Area of the semicircle $y = F(x) = (1 - x^2)^{1/2}$

Which of course we know is Pi/2 since it is the upper half of the circle, $x^2 + y^2 = 1$ with radius 1.

The parametric representation of this semicircle is

(cos(t),sin(t)) from 0 to Pi Thus,

Area = Integral –sin(t)sin(t) from 0 to Pi

WA 1 Integrate –sin(t)sin(t) from Pi to 0

Note the order of the limits since we want to go from -1 to 1 for a positive area.

Also, note that WA makes things even easier. We don't have to actually take the derivative as in WA 1

WA 2 integrate (derivative cos(t))*sin(t) from Pi to 0

Example 2. What is the area under one period of the cycloid? Plot it in case you forgot.

WA 3 Parametric Plot (t – sin(t), (1-cos(t)) from t = 0 to 2Pi

What is its area for one cycle?

WA 4 Integrate (Derivative t – sin(t))*(1-cos(t)) from t = 0 to 2Pi

How about that? 3 times the area of the circle.

Just remember, the graph can not fold back on itself.

That means: If $f(t_1) = f(t_2)$, then $t_1 = t_2$

Then the area under the curve (f(a),g(a) to (f(b),g(b)

Is Integral f'(t)g(t)dt from a to b

So, you can do a bunch of examples very easily using WA.

Integrate (Derivative f(t))*g(t) from t = a to b

Example 3. Find the area under the parabola $y = x^2$ from -2 to 2 using the parametric representation.

WA5 Parametric Plot (t,t^2) from t = -2 to 2

WA6 Integrate (derivative t)*(t^2) from t = -2 to 2

The non-parametric way

WA7 integrate x^2 from x = -2 to 2

Example 4. Find the area under a semi-ellipse defined by parametrically by (7cos(t), 3sin(t)) with t from 0 to Pi

WA8 Parametric Plot (7cos(t), 3sin(t)) from t = 0 to Pi

WA9 Integrate(Derivative 7cos(t))*3sin(t) from t=Pi to 0

Note: This is also the ellipse defined by $x^2/49 + y^2/9 = 1$

So the semi-ellipse is graph of $y = 3(1 - x^2/49)^{1/2}$

WA10 integrate 3((1-(x^2))/49)^.5 from -7 to 7

Example 5 curve defined parametrically by

(t^2+1 ,-cos(t)+.1sin(50t)+1), t=1 to 2.5

WA11

Parametric Plot (t^2+1 ,-cos(t)+.1sin(50t)+1), t=1 to 2.5

WA12

Integrate (Derivative (t^2+1))*(- cos(t)+.1sin(50t)+1) from t=1 to 2.5

WA13

Integrate 2t*(- cos(t)+.1sin(50t)+1) from t=1 to 2.5

Could you do this by creating the function F(x)?

Try it.

Yes

The curve is defined parametrically by

(t^2+1 ,-cos(t)+.1sin(50t)+1), t=1 to 2.5

Let $x = t^2+1$ and thus $t = (x - 1)^{1/2}$

y = -cos(t)+.1sin(50t)+1

We want y = F(x)

Substitute for t and evaluate the x limits.

$1^2+1 = 2$, and $(2.5)^2 +1 = 7.25$

$F(x) = 1 - \cos((x - 1)^{1/2}) + .1\sin(50(x - 1)^{1/2})$ from x = 2 to 7.25

WA14

Integrate 1 – cos((x-1)^.5) + .1sin(50(x-1)^.5) from x = 2 to 7.25

Of course, same answer.

T5 C24 Parametric Functions – Area Exercises

Q1. Find the area under the parabola $y = x^4$ from -3 to 3 using the parametric representation.

Q2. Find the area under the parabola $y = x^2 + x$ from -5 to 5 using the parametric representation.

Q3. Find the area of the upper semicircle for the function $x^2 + y^2 = 25$ using the parametric representation.

Q4. Find the area of the upper semicircle for the function $x^2 + y^2 = 49$ using the parametric representation.

Q5. Find the area of the upper semicircle for the function $x^2 + y^2 = 4$ using the parametric representation.

Q6. Find the area under a semi-ellipse defined parametrically by (2cos(t), 7sin(t)) with t from 0 to Pi

Q7. Find the area of the upper semi-ellipse for the function Note: This is also the ellipse defined by $x^2/36 + y^2/9 = 1$ using the parametric representation.

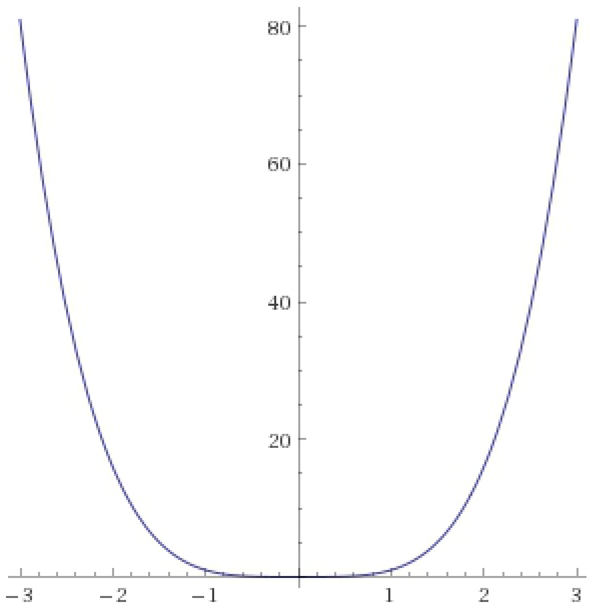
Q8. Find the area of the upper semi-ellipse for the function Note: This is also the ellipse defined by $x^2/4 + y^2/81 = 1$ using the parametric representation.

Q9. Find the area for a curve defined parametrically by $(t^3-2, -\sin(t)+0.5\sin(9t)+3)$, t=1 to 3.

Q10. Find the area for a curve defined parametrically by $(3t^3+6, -\cos(t)+0.7\sin(21t)+8)$, t=1 to 5.

A1. WA parametric plot (t, t^4) from t = -3 to 3

Parametric plot:



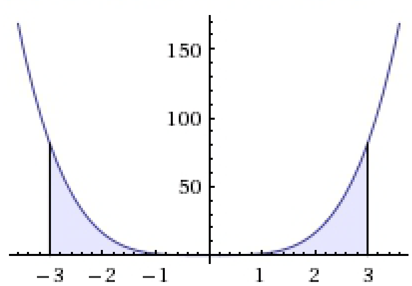
(t from −3 to 3)

WA Integrate (derivative t)*(t^4) from t = -3 to 3

Definite integral:

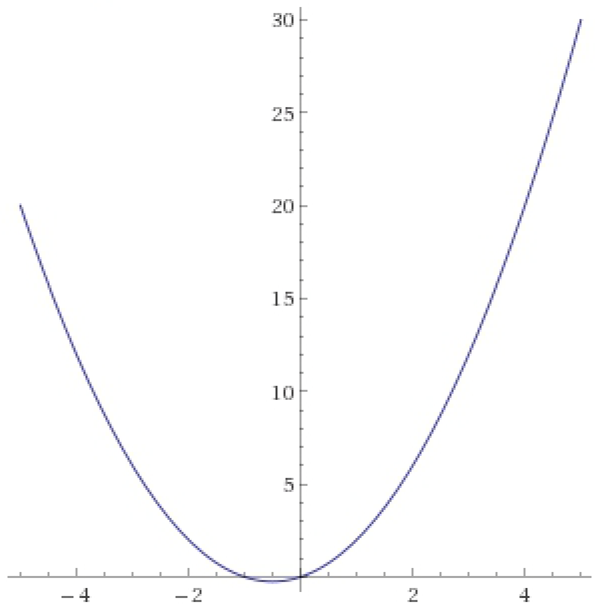
$$\int_{-3}^{3} t'(t)\, t^4 \, dt = \frac{486}{5} \approx 97.200$$

Visual representation of the integral:



A2. WA parametric plot (t,t^2 + t) from t = -5 to 5

Parametric plot:



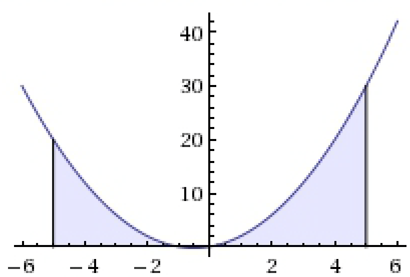
(t from −5 to 5)

WA Integrate (derivative t)*(t^2 + t) from t = -5 to 5

Definite integral:

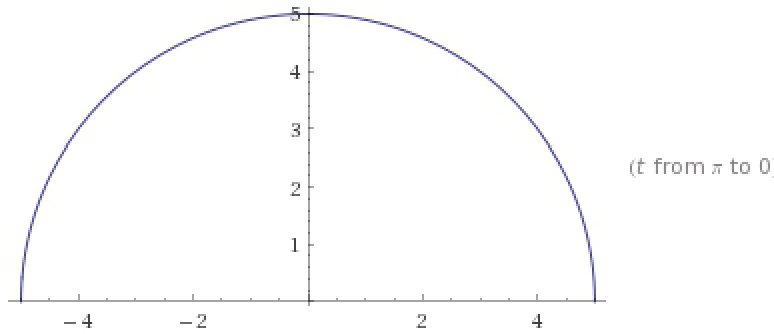
$$\int_{-5}^{5} t'(t)\left(t^2+t\right)dt = \frac{250}{3} \approx 83.333$$

Visual representation of the integral:



A3. WA parametric plot (5cos(t),5sin(t)) from Pi to 0

Parametric plot:

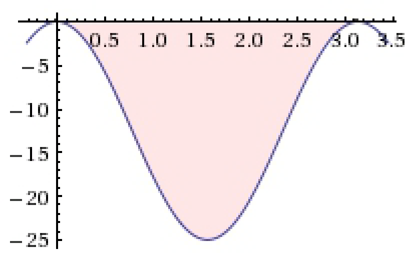


WA integrate (derivative 5cos(t))*5sin(t) from Pi to 0

Definite integral:

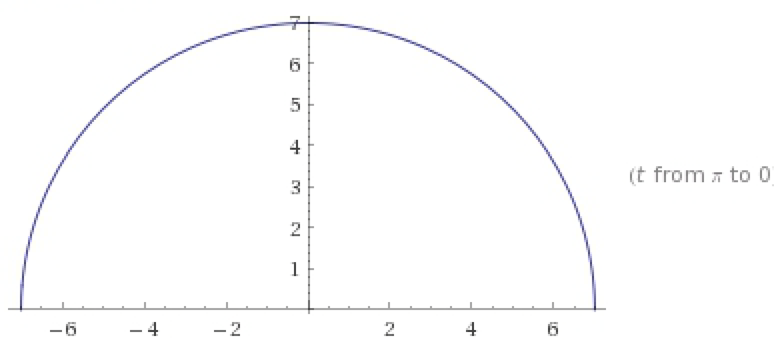
$$\int_{\pi}^{0} \frac{\partial(5\cos(t))}{\partial t} (5\sin(t))\, dt = \frac{25\,\pi}{2} \approx 39.270$$

Visual representation of the integral:



A4. WA parametric plot (7cos(t),7sin(t)) from Pi to 0

Parametric plot:

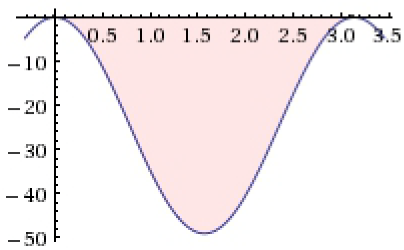


WA integrate (derivative 7cos(t))*7sin(t) from Pi to 0

Definite integral:

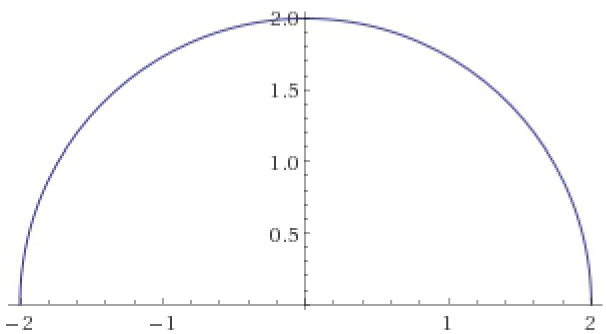
$$\int_{\pi}^{0} \frac{\partial(7\cos(t))}{\partial t} (7\sin(t))\, dt = \frac{49\,\pi}{2} \approx 76.969$$

Visual representation of the integral:



A5. WA parametric plot (2cos(t),2sin(t)) from Pi to 0

Parametric plot:



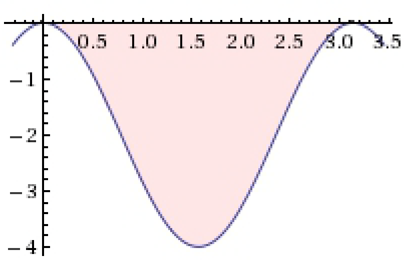
(t from π to 0)

WA integrate (derivative 2cos(t))*2sin(t) from Pi to 0

Definite integral:

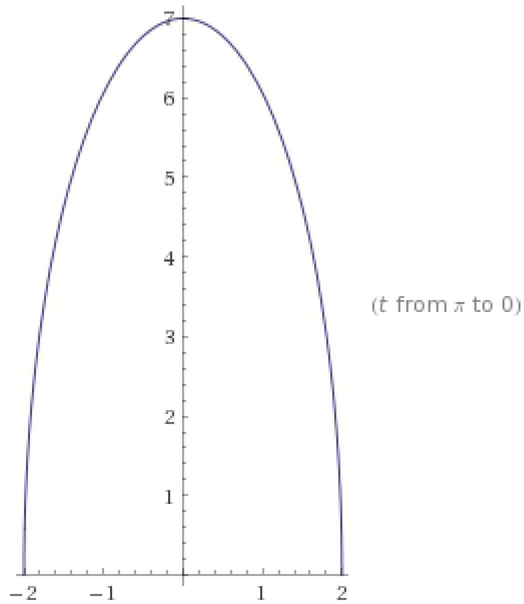
$$\int_{\pi}^{0} \frac{\partial(2\cos(t))}{\partial t} (2\sin(t))\, dt = 2\,\pi \approx 6.2832$$

Visual representation of the integral:



A6. WA parametric plot (2cos(t),7sin(t)) from Pi to 0

Parametric plot:

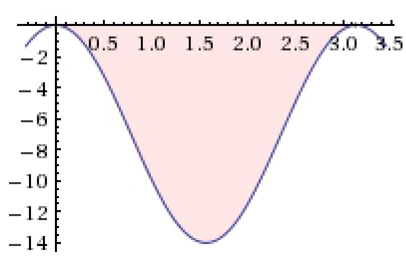


WA integrate (derivative 2cos(t))*7sin(t) from Pi to 0

Definite integral:

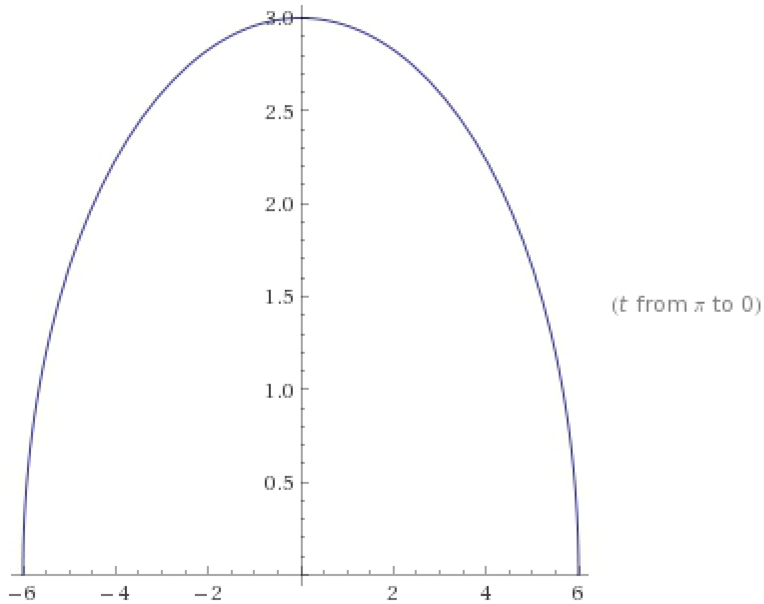
$$\int_{\pi}^{0} \frac{\partial(2\cos(t))}{\partial t}\,(7\sin(t))\,dt = 7\pi \approx 21.991$$

Visual representation of the integral:



A7. WA parametric plot (6cos(t),3sin(t)) from Pi to 0

Parametric plot:

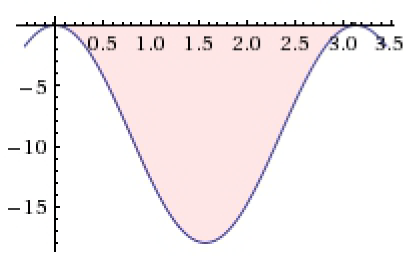


WA integrate (derivative 6cos(t))*3sin(t) from Pi to 0

Definite integral:

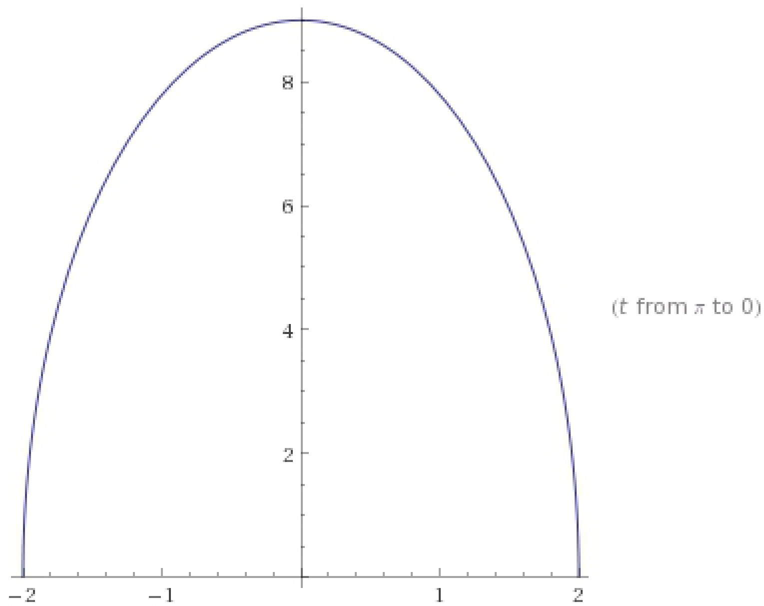
$$\int_{\pi}^{0} \frac{\partial(6\cos(t))}{\partial t}\,(3\sin(t))\,dt = 9\pi \approx 28.274$$

Visual representation of the integral:



A8. WA parametric plot (2cos(t),9sin(t)) from Pi to 0

Parametric plot:

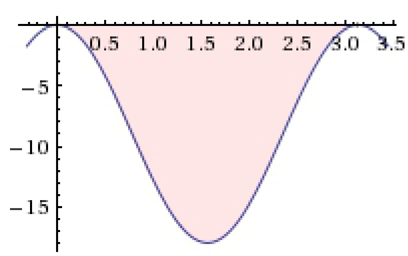


WA integrate (derivative 2cos(t))*9sin(t) from Pi to 0

Definite integral:

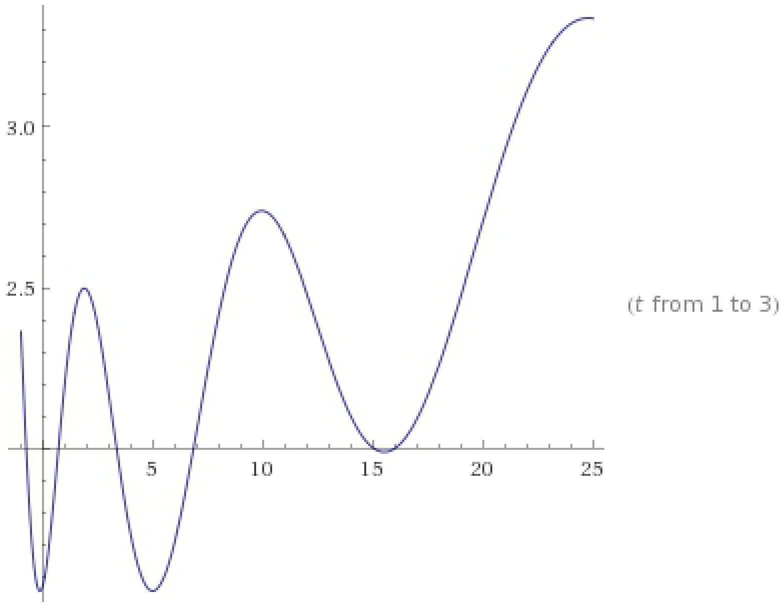
$$\int_{\pi}^{0} \frac{\partial(2\cos(t))}{\partial t}(9\sin(t))\,dt = 9\pi \approx 28.274$$

Visual representation of the integral:



A9. WA Parametric Plot (t^3-2 ,-sin(t)+0.5sin(9t)+3), t=1 to 3

Parametric plot:

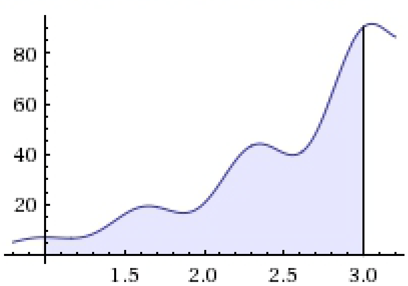


WA Integrate (3t^2)*(-sin(t)+0.5sin(9t)+3) from t=1 to 3

Definite integral:

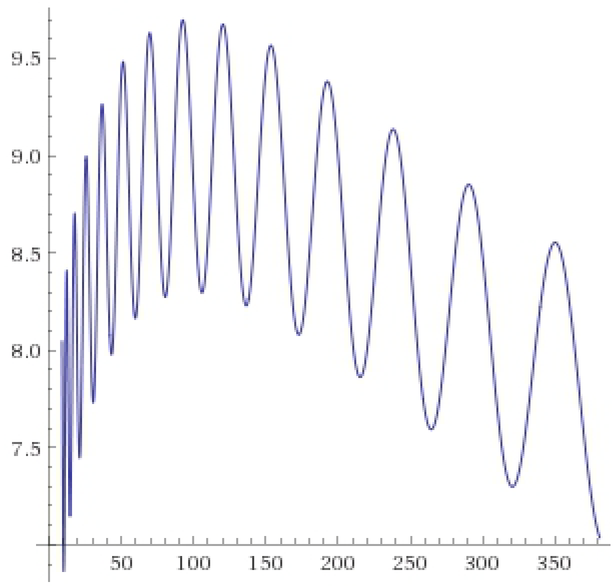
$$\int_1^3 (3\,t^2)\,(-\sin(t) + 0.5\,\sin(9\,t) + 3)\,dt = 61.7196$$

Visual representation of the integral:



A10. WA Parametric Plot (3t^3+6 ,-cos(t)+0.7sin(21t)+8), t=1 to 5

Parametric plot:



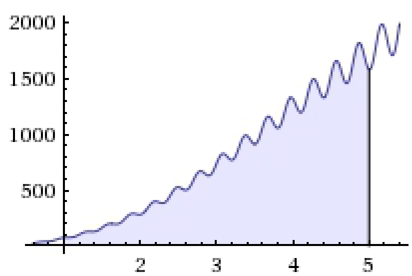
(t from 1 to 5)

WA Integrate (9t^2)*(-cos(t)+0.7sin(21t)+8) from t=1 to 5

Definite integral:

$$\int_1^5 (9\,t^2)\,(-\cos(t) + 0.7\sin(21\,t) + 8)\,dt = 3152.6$$

Visual representation of the integral:



T5 C25 Wolfram Alpha Commands

In Tier 5 you have learned to use many Wolfram Alpha commands to solve problems using calculus concepts.

This is a list of those commands.

You may find explanations and examples in the appropriate lesson.

These are in the order they were introduced in Tier 5.

I. Graphing a function, f(x), Lessons C2 and C2a

WA1 Plot f(x)

WA2 Plot f(x) from a to b

WA3 Roots f(x)

WA4 Stationary Points f(x)

WA5 Inflection Points f(x)

WA6 Asymptotes f(x)

Plus three more covered later in Part 1

WA8 Plot f(x,y) = 0 from x = a to b (T5 C8)

WA9 Solve for y when x = a, f(x,y) = 0

WA10 Tangent line when x = a, f(x,y) = 0

You will note that no calculus is needed to understand and apply these Graphing commands.

The Graph of a function is probably the best way to understand its behavior.

II. Part 1 Differentiation Lessons C3 thru C12

WA7 Derivative f(x)

WA8 Plot f(x,y) = 0 from x = a to b (T5 C8)

WA9 Solve for y when x = a, f(x,y) = 0

WA10 Tangent line when x = a, f(x,y) = 0

WA11 Solve for x, f(x,a)

WA12 Derivative f(x,y) = 0

WA13 g(x,y) at x = a and y = b

WA14 derivative inverse f(x) (T5 C10)

WA15 series f(x) (T5 C11)

WA16 series f(x) at x = a

WA17 series f(x) at x = a order n

III. Part 2 Integration Lessons C13 thru C24

WA18 antiderivative f(x)

WA18a indefinite integral f(x)

WA18b integrate f(x)

WA19 integrate f(x) from a to b

WA20 integrate absolute value f(x) from a to b

WA21 area f(x) from a to b

WA22 arc length f(x) from a to b

WA23 volume of revolution f(x) from x= a to b

WA24 volume of solid of revolution of region between f(x) and g(x) from a to b about the x axis.

WA25 surface of revolution f(x) from x = a to b

WA26 Parametric Plot (f(t),g(t))

WA27 Parametric Plot (f(t),g(t)) from t = a to b

WA28 Arc Length (f(t),g(t)) from t = a to b

WA29

y – g(t) = ((Derivative g(t)) / (Derivative f(t)))(x-f(t)), t=a

This is the tangent line to (f)t),g(t)) at t = a

WA30 Integrate (Derivative f(t))*g(t)) from t = a to b

This is area under the Parametrically defined curve (f(t),g(t) from t = a to b, assuming no fold back.

T5 C26 Improper Integrals Vertical Asymptotes

Definition of Definite Integral.

Let f(x) be a function defined and continuous on the closed interval [a,b]. Then you have learned that the definite integral is defined to be the area under the graph of this function from a to b.

And you have learned that the Fundamental Theorem of Calculus (FTC) is how you can evaluate the Integral using antiderivatives.

Of course, the tool Wolfram Alpha, WA, is a modern way of doing this automatically and most STEM professionals use such a tool, and not the old manual integration techniques our ancestors used for three centuries until very modern times.

It is important that f(x) is <u>continuous on the closed interval [a, b]</u>. Then it can be proven pretty easily that the graph of f(x) is bounded and that the definite integral is well defined and finite.

Now, what if f(x) is not continuous at all points in [a,b]?

In particular, suppose f(x) has a vertical asymptote at a point c where a < c < b. Could we define and evaluate a definite integral in this case?

The answer is yes, sometimes.

If you are a future math major, you might want to think how this might be done.

Example 1. Suppose f(x) = 1/abs(x) from -1 to 1

WA 1 Plot abs(x) from -1 to 1

WA 2 Plot 1/abs(x) from -1 to 1

Do you think the integral from -1 to 1 exists? After all the area extends to infinity around x = 0. Certainly from what we have learned so far the FTC does not apply.

WA 3 Integral 1/abs(x) from -1 to 1

Well, WA says "Integral does not converge". So, NO.

Example 2. Suppose f(x) = 1/(abs(x))^2 from -1 to 1

What do you think?

WA 4 Integral 1/(abs(x))^2 from -1 to 1

No again.

Example 3. Suppose f(x) = 1/(abs(x))$^{1/2}$ from -1 to 1

What do you think?

Well, let's ask WA again.

WA 5 Integral 1/(abs(x))^.5 from -1 to 1

Oh, WOW. WA says it does exist and equals what?

Answer: 4.

Even though this area is also infinite in extent it is somehow finite!

How can this be?

How could this integral be defined, and calculated?

Let's examine Examples 2 and 3 more closely and see how this can be handled.

Example 2. Suppose $f(x) = 1/(\text{abs}(x))^2$ from -1 to 1

First let's find an antiderivative.

WA 6 Integral 1/(abs(x))^2

Answer: $-1/x$

Now let's just look at the definite integral from .1 to 1

We know by the FTC that it will be $-1 + 10 = 9$

WA 7 Integral 1/(abs(x))^2 from .1 to 1

Let's get closer to 0 with the lower limit

WA 8 Integral 1/(abs(x))^2 from .001 to 1

By the FTC we know the answer will be $-1 + 1000 = 999$

What will be the integral from c to 1 for $0 < c < 1$?

$-1 + 1/c$ by the FTC.

What will the limit of this be when $c \rightarrow 0$?

Clearly, infinity.

So, if we define the Improper Integral of f(x) from 0 to 1 to be the limit of the Integral of f(x) from c to 1 when $c \rightarrow 0$ then in this example, the Improper Integral is infinite.

Exercise: You work out Example 3.

Definition: if $f(x)$ has a vertical asymptote at $x = c < b$ then we define the Improper Integral of $f(x)$ from c to b as the limit of the definite integral from d to b as $d \to c$

We assume $f(x)$ is continuous for all $c < x \leq b$ so we can use the FTC. You write this down

Example 3. Suppose $f(x) = 1/(\text{abs}(x))^{1/2}$ from -1 to 1

Now, we will just look at the improper integral from 0 to 1.

WA 9 Integral 1/(abs(x))^.5

Well, this answer is a little complicated since we are using the absolute value of x. But, this will just be x for $x>0$.

WA10 Integral 1/x^.5

Ans: $2x^{.5}$

So, by FTC the integral from c to 1 is: $2 - 2c^{.5}$

Thus, the Improper Integral from 0 to 1 is = 2.

WA11 Integral 1/x^.5 from 0 to 1

Due to symmetry we see the integral from -1 to 1 is just 4.

Of course, WA will let us do problems that were very difficult with classical calculus.

Furthermore, the definition for an improper integral coming from the left to an asymptote at $x = c > a$, is just the limit of the definite integral from a to d as $d \to c$

And, if $a < c < b$ and $x = c$ is a vertical asymptote, then the improper integral from a to b is just the sum of the two improper integrals just defined, one from a to c, and two from c to b.

Example 4

WA 12 Integral 1/(sin(x)) from 0 to 1

Infinite. No big surprise. But what about

WA 13 Integral 1/(sin(x))^.5 from 0 to 1

Ans: 2.03481

This would have been a very difficult problem with classical calculus techniques of integration. Note the answer is from antiderivative elliptic integrals.

So, if you are interested in a function, first plot it.

Determine where the vertical asymptotes are, if any.

Then, if you need a definite integral spanning any of the vertical asymptotes you will be evaluating improper integrals.

Of course, WA does this for you automatically.

Note: Integral of $f(x) / (x - c)^n$ from c to b will be

Infinite if $n \geq 1$

Finite if $0 < n < 1$

Example 5 integral $e^x / (x - 2)^{1.5}$ from 2 to 3

WA 14 integral e^x / (x – 2)^1.5 from 2 to 3

Example 6 integral $e^x / (x - 2)^{.7}$ from 2 to 3

WA 15 integral e^x / (x – 2)^.7 from 2 to 3

In the next lesson we will look at improper integrals of functions which have y = 0 as a horizontal asymptote.

You might try to see if you can see how to define such an improper integral and evaluate it. You should be able to do this with WA help.

T5 C26 Improper Integrals Vertical Asymptotes Exercises

Q1. Given the rules for the general setup $f(x)/(x-c)^n$, where will the graph show a vertical asymptote for the following function $x^2/(x-5)^2$? Will the integral be infinite or finite?

Q2. Given the rules for the general setup $f(x)/(x-c)^n$, where will the graph show a vertical asymptote for the following function $x^3/(x+1)^{0.2}$? Will the integral be infinite or finite?

Q3. Given the rules for the general setup $f(x)/(x-c)^n$, where will the graph show a vertical asymptote for the following function $(x-4)^2/(x-5)^3$? Will the integral be infinite or finite?

Q4. Where will the graph show a vertical asymptote for the following function $2x^3/(x-3)^{0.2}$? Plot the function to verify. Will the integral be infinite or finite? If finite, what is the value of the integral from x = 0 to 6?

Q5. Where will the graph show a vertical asymptote for the following function $e^x/(x+2)^{0.6}$? Plot the function to verify. Will the integral be infinite or finite? If finite, what is the value of the integral from x = -4 to 0?

Q6. Where will the graph show a vertical asymptote for the following function $x/(x+5)^2$? Plot the function to verify. Will the integral be infinite or finite? If finite, what is the value of the integral from x = -6 to -3?

Q7. Where will the graph show a vertical asymptote for the following function $(x+3)^2/(x-2)^{0.8}$? Plot the function to verify. Will the integral be infinite or finite? If finite, what is the value of the integral from x = 0 to 5?

Q8. Where will the graph show a vertical asymptote for the following function $\ln(x)/(2x+4)^{1.5}$? Plot the function to verify. Will the integral be infinite or finite? If finite, what is the value of the integral from x = -4 to -0?

Q9. Where will the graph show a vertical asymptote for the following function $e^x/(2x-9)^{0.3}$? Plot the function to verify. Will the integral be infinite or finite? If finite, what is the value of the integral from x = 2 to 6?

Q10. Where will the graph show a vertical asymptote for the following function $\sin(x)/(3x-9)^2$? Plot the function to verify. Will the integral be infinite or finite? If finite, what is the value of the integral from x = 1 to 4?

A1. Vertical asymptote: 5; infinite

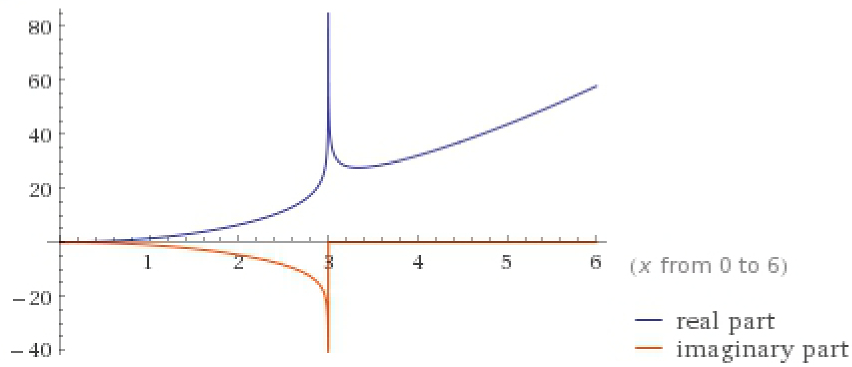
A2. Vertical asymptote: -1; finite

A3. Vertical asymptote: 5; infinite

A4. Vertical asymptote: 3; finite

WA Plot (2x^2)/(x - 3)^(0.2) from 0 to 6

Plot:



WA Integral (2x^2)/(x - 3)^(0.2) from 0 to 6

Definite integral:

$$\int_0^6 \frac{2x^2}{(x-3)^{0.2}}\,dx = -\frac{5}{14}\,3^{4/5}\left(25\,(-1)^{4/5} - 137\right) \approx 135.226 - 12.6386\,i$$

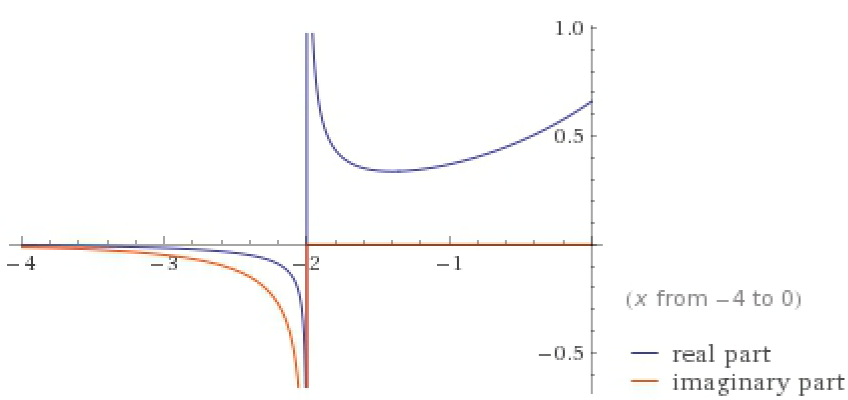
Indefinite integral:

$$\int \frac{2x^2}{(x-3)^{0.2}}\,dx = \frac{5}{42}\,(x-3)^{4/5}\left(6x^2 + 20x + 75\right) + \text{constant}$$

A5. Vertical asymptote: -2; finite

WA plot e^x/(x + 2)^(0.6) from -4 to 0

Plot:



WA integral e^x/(x + 2)^(0.6) from -4 to 0

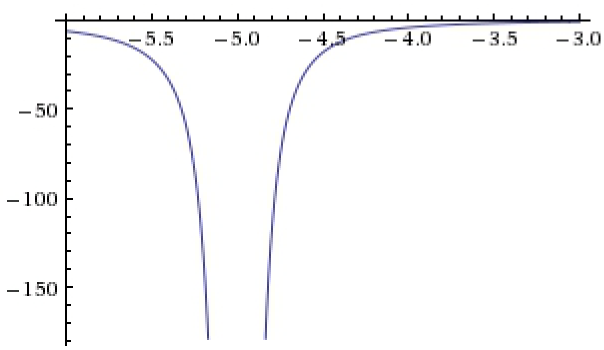
Definite integral:

$$\int_{-4}^{0} \frac{e^x}{(x+2)^{0.6}}\, dx = \frac{(-1)^{2/5}\left(-\left(1+\sqrt[5]{-1}\right)\Gamma\left(\frac{2}{5}\right)+\sqrt[5]{-1}\ \Gamma\left(\frac{2}{5}, -2\right)+\Gamma\left(\frac{2}{5}, 2\right)\right)}{e^2} \approx 0.86981 - 0.276123\, i$$

A6. Vertical asymptote: -5; infinite

WA plot x/(x + 5)^2 from -6 to -3

Plot:

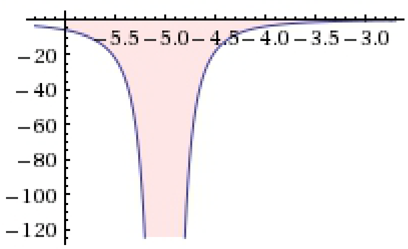


WA integral x/(x + 5)^2 from -6 to -3

Result:

(integral does not converge)

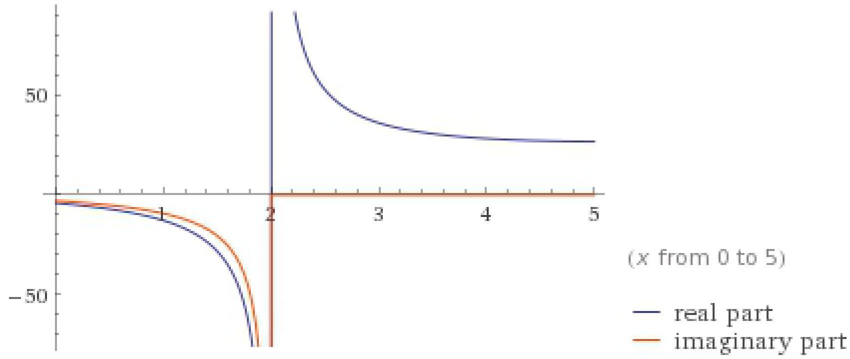
Visual representation of the integral:



A7. Vertical asymptote: 2; finite

WA plot (x + 3)^2/(x - 2)^(0.8) from 0 to 5

Plot:



WA integral (x + 3)^2/(x - 2)^(0.8) from 0 to 5

Definite integral:

$$\int_0^5 \frac{(x+3)^2}{(x-2)^{0.8}}\,dx = 88.8701 - 72.8364\,i$$

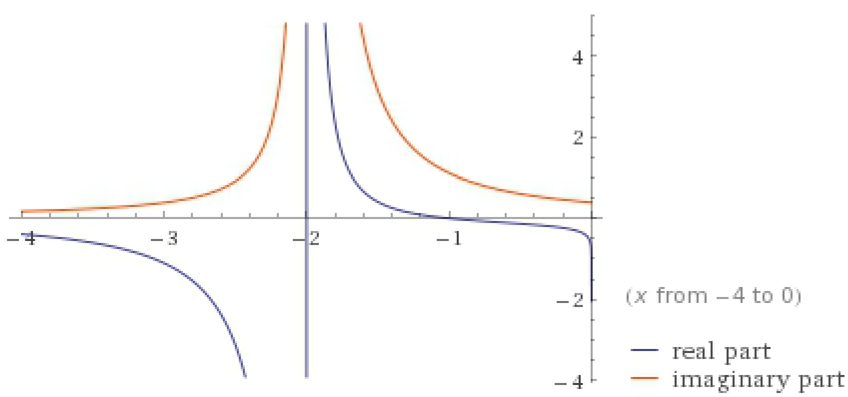
Indefinite integral:

$$\int \frac{(x+3)^2}{(x-2)^{0.8}}\,dx = \frac{5}{33}\sqrt[5]{x-2}\,\left(3x^2 + 43x + 727\right) + \text{constant}$$

A8. Vertical asymptote: -2; infinite

WA plot ln(x)/(2x + 4)^(1.5) from -4 to 0

Plot:



WA integral ln(x)/(2x + 4)^(1.5) from -4 to 0

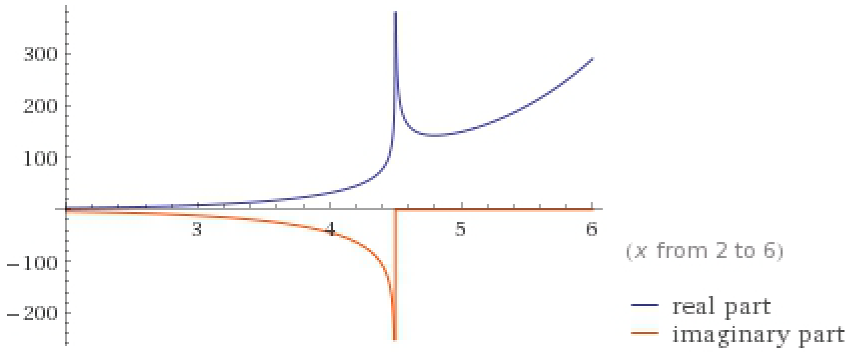
Result:

(integral does not converge)

A9. Vertical asymptote: 4.5; finite

WA plot e^x/(2x - 9)^(0.3) from 2 to 6

Plot:



WA integral e^x/(2x - 9)^(0.3) from 2 to 6

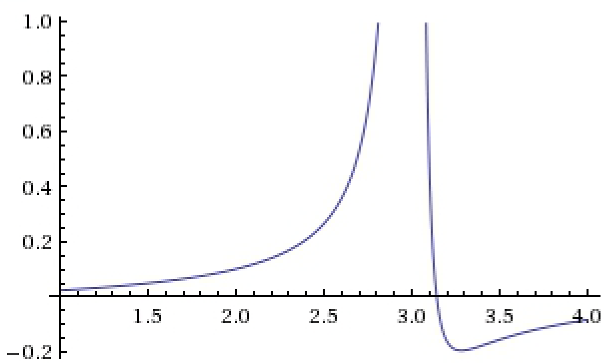
Definite integral:

$$\int_2^6 \frac{e^x}{(2x-9)^{0.3}}\,dx = \left(-\frac{1}{2}\right)^{3/10} e^{9/2}\left(-\left(1+(-1)^{2/5}\right)\Gamma\left(\frac{7}{10}\right)+\Gamma\left(\frac{7}{10},-\frac{3}{2}\right)+(-1)^{2/5}\,\Gamma\left(\frac{7}{10},\frac{5}{2}\right)\right) \approx 338.7 - 73.4069\,i$$

A10. Vertical asymptote: 3; infinite

WA plot sin(x)/(3x - 9)^2 from 1 to 4

Plot:

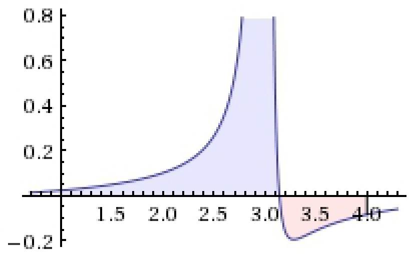


WA integral sin(x)/(3x - 9)^2 from 1 to 4

Result:

(integral does not converge)

Visual representation of the integral:



T5 C27 Improper Integrals Horizontal Asymptotes

Question: Do you think it would be theoretically possible to paint an infinite surface area with a finite volume of paint? Answer below.

Suppose we have a function f(x) which is continuous for all x ≥ a and which has the x-axis as a horizontal asymptote.

Clearly we can calculate the integral of f(x) from a to b for any b > a by the FTC.

Let us define the integral of f(x) from a to infinity as follows:

Sometimes this limit will exist and sometimes not.

This is called an Improper Integral.

We can now use Wolfram Alpha, WA, to evaluate such integrals directly.

Of course, you can do all of this going in the negative direction to –infinity.

Example 1. $f(x) = 1/x$

WA 1 integrate 1/x from 1 to inf

It is infinite. Not too surprising since we know the infinite sum 1/n is infinite.

WA 2 sum 1/n from 1 to inf

What would happen if we raise x to an exponent greater than 1?

Example 2 $f(x) = 1/x^{1.05}$

WA 3 Plot 1/x, 1/x^1.05 from 1 to 5

WA 4 integrate 1/x^1.05 from 1 to inf

Wow. Ans: 20

Let's verify these examples with the FTC and limit definition of improper integral;

WA 5 integrate 1/x

Ans: log(x)

So by FTC the integral of 1/x from 1 to b is log(b)

Thus, the limit as b → inf is infinity

WA 6 integrate 1 / x^1.05

Ans: $-20 / x^{1/20}$

So by FTC the integral of $1 / x^{1.05}$ from 1 to b

$-20 / b^{1/20} - (-20)$

Thus, the limit as b → inf is 20

We can now easily see that the integral from a to inf of $1 / x^n$ will be finite for any n > 1 and a>0 and will be infinite for any n ≤ 1

Example 3: integral of e^x from –inf to 0

WA 7 integrate e^x from 0 to –inf

Ans: -1

Oops. Reverse the order of limits of integration

WA 8 integrate e^x from -inf to 0

Ans: 1

Example 4. f(x) = which is useful in many STEM subjects.

WA 9 Plot e^-(x^2) from 0 to 5

The classical way is to find an antiderivative and then apply the FTC and take a limit

WA 10 integrate e^-(x^2)

The antiderivative is a special function. So this is not easy to deal with directly.

WA 11 integrate e^-(x^2) from 0 to inf

Ans: Square Root of Pi/2 = .886

Example 5. We know the area under the hyperbola y = 1/x from 1 to inf is infinite from WA 1 and our own analysis using the FTC and limit definition.

Suppose we rotate this hyperbola about the x-axis and create an infinite funnel from 1 to infinity. What would its volume be?

Believe it or not, for the first time I caught Wolfram Alpha in a mistake as you'll soon see.

I remembered that the improper integral of $1/x^2$ from 1 to infinity is 1.

WA 12 integrate 1/x^2 from 1 to inf

Ans: 1.

Think back to our lesson on volume of solid of revolution about the x axis.

The volume from a to b will be the integral of Pi / x^2 from a to b.

So, what about from 1 to 10

WA 13 volume of revolution of 1 / x from 1 to 10

Ans: 9Pi / 10

We could do this directly too.

WA 14 Integrate Pi / x ^ 2 from 1 to 10

OK, what about the volume of the solid of revolution from 1 to inf?

WA 15 volume of revolution of 1 / x from 1 to inf

Oops. Infinity?

It is always good to double check any result, if you can and if it is important.

This doesn't look right to me since I know the improper integral of 1 / x^2 is 1

WA 16 Integrate Pi / x ^ 2 from 1 to inf

Ans: Pi What is going on here?

WA 17 volume of revolution of 1/x from 1 to 1000

Ans: .999Pi

WA 18 volume of revolution of 1/x from 1 to 1000000

Ans: .999999Pi

WA 19 volume of revolution of 1/x from 1 to 10^10

Ans: .9999999999Pi

Do you see what is happening?

WA 20 volume of revolution of 1/x from 1 to 10^100

Ans: infinite. Clearly wrong. WA is somehow overloaded.

WA 21 integrate Pi/x^2 from 1 to 10^100

Ans: This is the correct answer which WA was able to calculate. Somehow their volume of revolution can't handle large limits including inf.

So theoretically, we have an infinite surface area embedded in a finite volume.

Of course, this math "model" could never be realized in the real world.

We always have to be very careful of our idealized math models in STEM subjects. They can be misleading sometimes.

T5 C27 Improper Integrals Horizontal Asymptotes Exercises

Q1. Given the function $1/x^{1.5}$, will the integral be finite or infinite?

Q2. Given the function $1/x^{0.5}$, will the integral be finite or infinite?

Q3. Given the function $1/x^{2.4}$, will the integral be finite or infinite? If it is finite, what is the value of the integral from 1 to infinity?

Q4. Given the function $1/x^{0.6}$, will the integral be finite or infinite? If it is finite, what is the value of the integral from 1 to infinity?

Q5. Given the function $1/x^{1.6}$, will the integral be finite or infinite? If it is finite, what is the value of the integral from 1 to infinity?

Q6. Given the function $\log(x)/x^{1.07}$, will the integral be finite or infinite? If it is finite, what is the value of the integral from 1 to infinity?

Q7. Given the function $e^{x}/x^{1.07}$, will the integral be finite or infinite? If it is finite, what is the value of the integral from 1 to infinity?

Q8. Given the function $\sin(x)/x^{1.07}$, will the integral be finite or infinite? If it is finite, what is the value of the integral from 1 to infinity?

Q9. Given the function $1/-x^{1.03}$, will the integral be finite or infinite? If it is finite, what is the value of the integral from -1 to negative infinity?

Q10. Given the function $1/-x^{0.4}$, will the integral be finite or infinite? If it is finite, what is the value of the integral from -1 to negative infinity?

A1. Finite; using the general formula, x^n, $n>1$

A2. Infinite; using the general formula, x^n, $n<1$

A3. Finite

WA integral 1/x^(2.4) from 1 to inf

Definite integral:

$$\int_1^\infty \frac{1}{x^{2.4}}\,dx = 0.714286$$

Indefinite integral:

$$\int \frac{1}{x^{2.4}}\,dx = -\frac{5}{7\,x^{7/5}} + \text{constant}$$

A4. Infinite

WA integral 1/x^(0.6) from 1 to inf

Input:

$$\int_1^\infty \frac{1}{x^{0.6}}\,dx$$

Result:

(integral does not converge)

A5. Finite

WA integral 1/x^(1.6) from 1 to inf

Definite integral:

$$\int_1^\infty \frac{1}{x^{1.6}}\,dx = 1.66667$$

Indefinite integral:

$$\int \frac{1}{x^{1.6}}\,dx = -\frac{5}{3\,x^{3/5}} + \text{constant}$$

A6. Finite

WA integral (log(x))/x^(1.07) from 1 to inf

Definite integral:

$$\int_1^\infty \frac{\log(x)}{x^{1.07}}\,dx = 204.082$$

Indefinite integral:

$$\int \frac{\log(x)}{x^{1.07}}\,dx = \frac{-14.2857\log(x) - 204.082}{x^{0.07}} + \text{constant}$$

A7. Infinite

WA integral (e^(x))/x^(1.07) from 1 to inf

Input:

$$\int_1^\infty \frac{e^x}{x^{1.07}}\,dx$$

Result:

(integral does not converge)

A8. Finite

WA integral (sin(x))/x^(1.07) from 1 to inf

Definite integral:

$$\int_1^\infty \frac{\sin(x)}{x^{1.07}}\,dx = 0.61905$$

A9. Finite

WA integral 1/-x^(1.03) from -inf to -1

Definite integral:

$$\int_{-\infty}^{-1} -\frac{1}{x^{1.03}}\,dx = 33.1854 - 3.13694\,i$$

Indefinite integral:

$$\int -\frac{1}{x^{1.03}}\,dx = \frac{33.3333}{x^{0.03}} + \text{constant}$$

A10. Infinite

WA integral 1/-x^(0.4) from -inf to -1

Input:

$$\int_{-\infty}^{-1} -\frac{1}{x^{0.4}}\,dx$$

Result:

(integral does not converge)

T5 C28 Improper Integrals Horizontal Asymptotes with Infinite Area Exercises

These exercises from C27 had integrals that did not converge. Will any of them have finite volume?

Q1. Given the function $1/x^{0.6}$, what is the value of the volume of revolution from 1 to infinity?

Q2. Given the function $e^x/x^{1.07}$, what is the value of the volume of revolution from 1 to infinity?

Q3. Given the function $1/-x^{0.4}$, what is the value of the volume of revolution from 1 to infinity?

A1. WA volume of revolution 1/x^(0.6) from 1 to inf

Result:

$$\int_1^\infty \frac{\pi}{x^{1.2}}\,dx = 15.708$$

A2. WA volume of revolution (e^(x))/x^(1.07) from 1 to inf

Result:

(integral does not converge)

WA volume of revolution (e^(x))/x^(1.07) from 1 to 10

Result:

$$\int_1^{10} \frac{e^{2x}\,\pi}{x^{2.14}}\,dx = 6.23102 \times 10^6$$

WA volume of revolution (e^(x))/x^(1.07) from 1 to 10^6

Result:

$$\int_1^{1000000} \frac{e^{2x}\,\pi}{x^{2.14}}\,dx = 2.088944229576165 \times 10^{868576}$$

A3. WA volume of revolution 1/-x^(0.4) from -inf to -1

Result:

(integral does not converge)

T5 C29 Applications of Integration Surface Area of solid of revolution

Let f(x) be a function defined with derivative over (a,b)

Rotate f(x) about the x-axis to form a solid of revolution

What is the surface area of this solid of revolution?

Slice the solid of revolution into very thin discs with thickness, dx.

Now, what would be the surface area of one such disc rotated about the x axis?

The arc length of the little portion of the disc from x to x+dx would be $[1 + (f'(x))^2]^{1/2}dx$ from what we learned in the Arc Length lesson.

The radius of this rotated portion would be f(x).

So the surface area would be

Definite Integral of $2\pi|f(x)|[1 + (f'(x))^2]^{1/2}dx$ from a to b

I use the absolute value of f(x) to account for the possibility it might be negative.

So, let's test this out with WA.

Example 1   f(x) = sin(x) from 0 to Pi

WA1 volume of solid of revolution of sin(x) about x axis from 0 to Pi

Answer: Volume = 4.9348   Surface Area = 14.4236

WA2 surface area of solid of revolution of sin(x) about x axis from 0 to Pi

Same answers.

So, we see you can get both answers two ways.

Of course, we could also simply construct the integrals and integrate them. This is how we did it in the old days before tools like WA. It is easy to set up the integral.

Finding the antiderivative to apply the Fundamental Theorem of Calculus to evaluate the definite integral was the hard part.

WA3 integrate 2Piabs(sin(x))(1 + cos^2(x))^.5

Note the answer for antiderivative.

This would have been very difficult to find using the standard techniques of integration taught in a typical calculus course even today.   Then, of course, one would have to evaluate this antiderivative at the two limit points, which in this example involves an inverse hyperbolic sin.

Note, WA gives both the antiderivative and evaluates it.

In more difficult cases WA simply evaluates the definite integral when no "simple" antiderivative is available.

Example 2.

WA4 volume of solid of revolution of x^.5 about x axis from 0 to 4

Answer: V = 25.1327 SA = 48.7433

Now, let's just do the surface area.

WA5 surface area of solid of revolution of x^.5 about x axis from 0 to 4

Answer: V = 25.1327 SA = 36.1769

Oops! What went wrong.

Pause the video and see if you can figure this out.

WA is a tool.

Sometimes, it doesn't work the way you thought it would.

You must use your head, and always think about the results.

When possible try to do the problem in more than one way.

WA6 integrate 2Pix^.5(1 + (.5x^-.5)^2)^.5 from 0 to 4

Ans: 36.1769

So, WA5 surface area was correct.

What went wrong with the WA4 volume command?

Hint: What is area of the end cap of this solid of revolution?

Answer: The radius of this end cap is $4^{1/2}$ or 2.

Thus, its area is $Pi2^2$ = 4Pi = 12.5664

And, 36.1769 + 12.5664 = 48.7433

Actually, we could have seen that from the volume answer if we looked closely.

Example 3

WA7 volume of solid of revolution of x^2 about x axis from .7 to 3.6

Answer: V = 379.815 SA = 1064.98

Do you believe SA answer?

What about the area of the two end caps?

WA8 surface area of solid of revolution of x^2 about x axis from .7 to 3.6

How would you have found this the classical way?

WA9 integrate 2Pi(x^2)(1+ 4x^2)^.5

Example 4

WA10 surface area of solid of revolution of x^2 + .3sin(10x) about x axis from 1 to 2

To do this classically you would have to find the antiderivative of:

WA11 integrate 2Pi(abs(x^2+.3sin(10x)))(1 +(1+(2x+3.cos^2(10x)))^.5

WA can't do it, because there is NO antiderivative from known functions. Neither can I.

In fact, many problems that arise in STEM subjects do not have any solution via the Fundamental Theorem because there is no antiderivative from known defined functions including the so called Special Functions.

So, what did our ancestors do before computers?

They expanded f(x) is a power series expansion using some technique like a Taylor expansion, and then approximated f(x) with a polynomial p(x).

This was not easy and very time consuming, but doable if one wanted the answer bad enough.

With a modern computer you just approximate the definite integral with many subdivisions. That is probably what WA does. It is very fast and easy with a properly programmed computer.

Exercise: Play with this. I will discuss it in C30

Find the region enclosed by x^2 and 3sin(x) and rotate it about the x axis and find its volume and surface area.

This will give you some good experience with WA and thinking about how to use it.

T5 C29 – Applications of Integration – Surface Area of Solid of Revolution Exercises

For each of the following functions in questions 1-10, use Wolfram Alpha to find the Volume of Solid of Revolution and the Surface Area of Solid of Revolution. If WA gives you differing answers, use the integration technique discussed in the video to determine which answer is correct. Also, find the area of any end caps to account for the discrepancy.

Q1. $f(x) = x^3$ from $x = 1$ to 2

Q2. $f(x) = \cos(x)$ from $x = \pi/2$ to $3\pi/2$

Q3. $f(x) = \sin^2(x)$ from $x = 0$ to π

Q4. $f(x) = \cos(x)$ from $x = 0$ to π

Q5. $f(x) = x^3 - 0.3x^2$ from $x = 0.1$ to 0.28

Q6. $f(x) = \sin^2(x) - \sin(3x)$ from $x = 2$ to 4

Q7. $f(x) = x^2 + 0.3\cos(8x)$ from $x = 0.6$ to 2.7

Q8. $f(x) = x^{0.5} + 0.2\sin(3x)$ from $x = 0$ to 3

Q9. $f(x) = 4x^2 - 36$ Find where $f(x) = 0$, and use these points as the boundaries for x

Q10. $f(x) = 0.5\sin(6x)$ Find where $f(x) = 0$ (starting with 0), use the first two points as the boundaries for x

For Q11 – Q18, use the "surface of solid of revolution of region" method to find the surface area of f(x) rotated about the x-axis.

Q11. $f(x) = x$ from $x = 0$ to pi

Q12. $f(x) = x^2$ from $x = 1$ to 4

Q13. $f(x) = e^x$ from $x = 0$ to 2

Q14. $f(x) = x^{1/2}$ from $x = 0$ to 1

Q15. $f(x) = \sin(x) + 3$ from $x = 0$ to 2pi

Q16. $f(x) = 3x + x^{1/2}$ from $x = 0$ to 3

Q17. $f(x) = 2\cos(3x) + 8$ from $x = 0$ 4pi

Q18. $f(x) = 2x^2 + 1$ from $x = 0$ to 2

Q19. Find the formula WA provided in the answer to the previous question (Q18). Which part of that formula for the surface area about the x-axis corresponds to the circumference of a disk?

Which part corresponds to the radius of that disk?

Q20. What would you change in our surface area formula if you wanted to find the surface area of a function rotated only halfway around the x-axis?

A1. WA volume of solid of revolution x^3 from 1 to 2

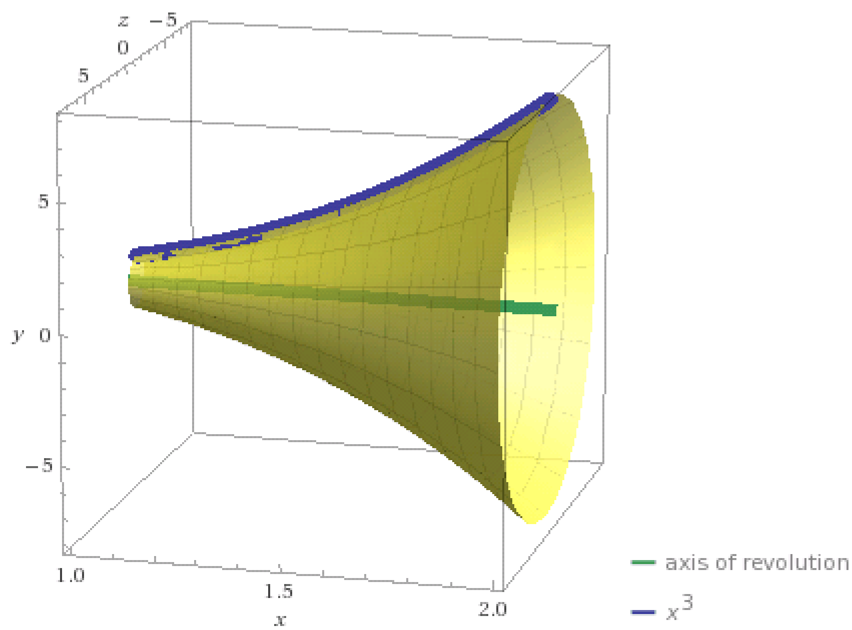
Result:

$$\int_1^2 \pi x^6 \, dx = \frac{127\pi}{7} \approx 56.9975$$

Surface area of solid:

$$65\pi - \frac{5}{27}\left(2\sqrt{10} - 29\sqrt{145}\right)\pi \approx 403.684$$

Plot:



WA surface area of solid of revolution x^3 from 1 to 2

Result:

$$\int_1^2 2\pi\sqrt{1+9x^4}\,|x|^3 \, dx = -\frac{5}{27}\left(2\sqrt{10} - 29\sqrt{145}\right)\pi \approx 199.48$$

Volume of solid:

$$\int_1^2 \pi x^6 \, dx = \frac{127\pi}{7} \approx 56.9975$$

Note the surface area discrepancy. Which is correct?

WA definite integral of 2piabs(x^3)[1 + (3x^2)^2]^(1/2) from 1 to 2

Definite integral:

More digits

$$\int_1^2 2\pi\left(|x^3|\sqrt{1+(3x^2)^2}\right)dx = -\frac{5}{27}\left(2\sqrt{10}-29\sqrt{145}\right)\pi \approx 199.48$$

Area at x = 1: $\pi r^2 = \pi(f(x))^2 = \pi(1^3)^2 = 3.14159$

Area at x = 2: $\pi(2^3)^2 = 201.06193$

403.684 – (3.142 + 201.062) = 199.48

A2. WA volume of solid of revolution cos(x) from pi/2 to 3pi/2

Result:

More digits

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \pi \cos^2(x)\, dx = \frac{\pi^2}{2} \approx 4.9348$$

Parametric representation of solid:

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ \rho\cos(\theta)\cos(x_0) \\ \rho\sin(\theta)\cos(x_0) \end{pmatrix} \text{ for } \frac{\pi}{2} < x_0 < \frac{3\pi}{2} \text{ and } 0 < \theta < 2\pi \text{ and } 0 < \rho < 1$$

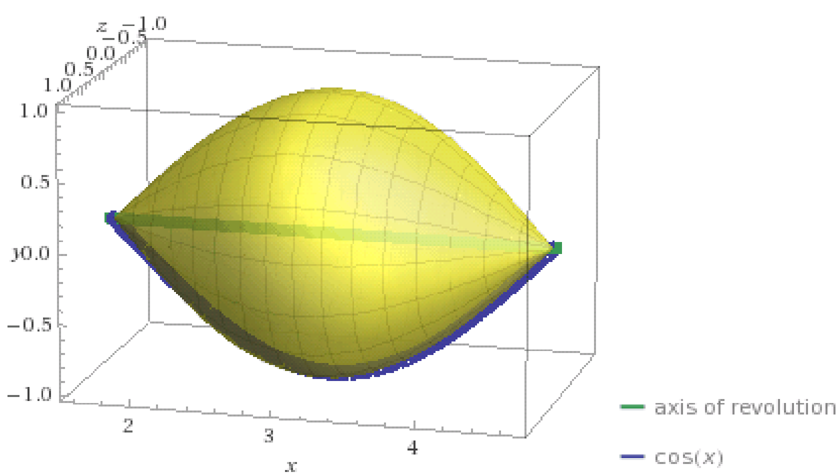
Surface area of solid:

More digits

$$2\pi\left(\sqrt{2} + \sinh^{-1}(1)\right) \approx 14.4236$$

$\sinh^{-1}(x)$ is the inverse hyperbolic sine function

Plot:



WA surface area of solid of revolution cos(x) from pi/2 to 3pi/2

Result:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} 2\pi |\cos(x)| \sqrt{1+\sin^2(x)}\, dx = 2\pi\left(\sqrt{2} + \sinh^{-1}(1)\right) \approx 14.4236$$

$|z|$ is the absolute value of z

Volume of solid:

$$\int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \pi \cos^2(x)\, dx = \frac{\pi^2}{2} \approx 4.9348$$

Note there is no surface area discrepancy. Why?

Compare the graphs in A1 and A2. The graph will be a clue as to whether or not there will be a discrepancy in the surface area value between the two methods.

A3. WA volume of solid of revolution (sin^2(x)) from 0 to pi

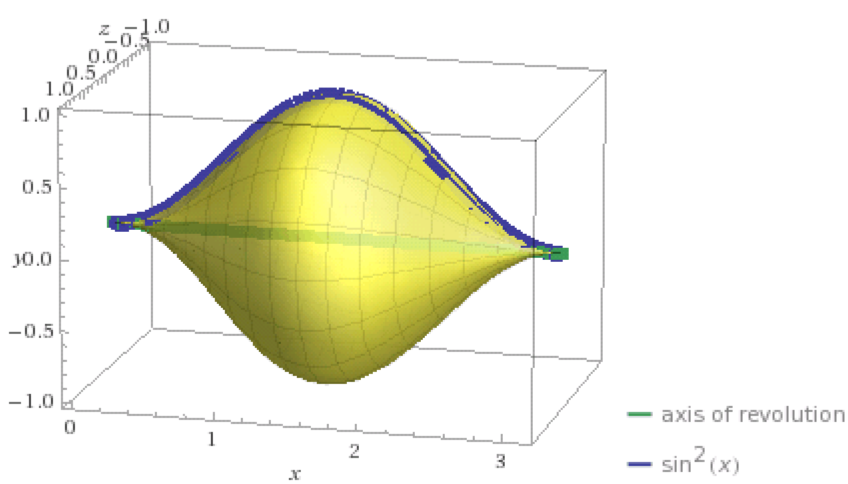
Result:

$$\int_0^{\pi} \pi \sin^4(x)\, dx = \frac{3\pi^2}{8} \approx 3.7011$$

Surface area of solid:

12.0015

Plot:



WA surface area of solid of revolution (sin^2(x)) from 0 to pi

Result:

$$\int_0^\pi 2\,\pi\,|\sin(x)|^2 \sqrt{1+\sin^2(2\,x)}\;dx = 12.0015$$

Volume of solid:

$$\int_0^\pi \pi \sin^4(x)\,dx = \frac{3\,\pi^2}{8} \approx 3.7011$$

A4. WA volume of solid of revolution (cos(x)) from 0 to pi

Result: More digits

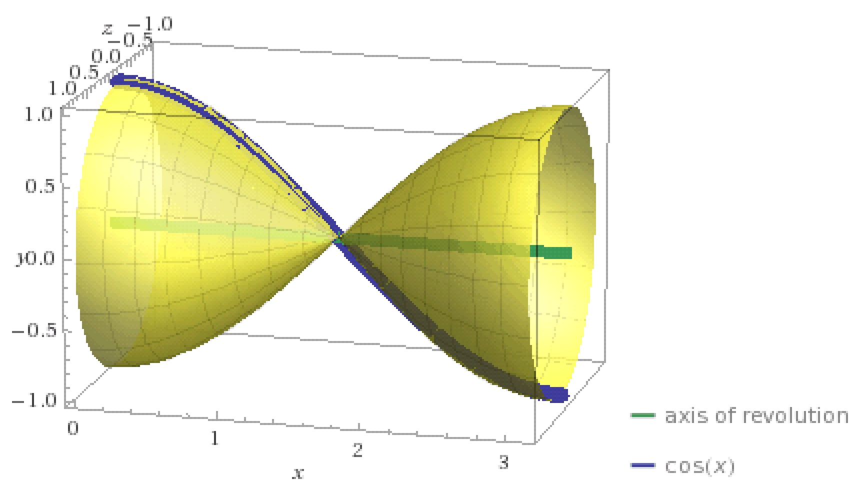
$$\int_0^\pi \pi \cos^2(x)\,dx = \frac{\pi^2}{2} \approx 4.9348$$

Surface area of solid: More digits

$$2\,\pi + 2\,\pi\left(\sqrt{2} + \sinh^{-1}(1)\right) \approx 20.7068$$

$\sinh^{-1}(x)$ is the inverse hyperbolic sine function

Plot: Show solid



WA surface area of solid of revolution (cos(x)) from 0 to pi

Result:

More digits

$$\int_0^{\pi} 2\pi\, |\cos(x)| \sqrt{1+\sin^2(x)}\, dx = 2\pi\left(\sqrt{2}+\sinh^{-1}(1)\right) \approx 14.4236$$

$|z|$ is the absolute value of z

$\sinh^{-1}(x)$ is the inverse hyperbolic sine function

Volume of solid:

More digits

$$\int_0^{\pi} \pi \cos^2(x)\, dx = \frac{\pi^2}{2} \approx 4.9348$$

WA definite integral of 2piabs(cos(x))[1 + (-sin(x))^2]^(1/2) from 0 to pi

Definite integral:

More digits

$$\int_0^{\pi} 2\pi\left(|\cos(x)| \sqrt{1+(-\sin(x))^2}\right) dx = 2\pi\left(\sqrt{2}+\sinh^{-1}(1)\right) \approx 14.424$$

Area at x = 1: $\pi r^2 = \pi(f(x))^2 = \pi(\cos(0))^2 = 3.14159$

Area at x = 2: $\pi(\cos(\pi))^2 = 3.14159$

20.7068 – (3.1416 + 3.1416) = 14.4236

A5. WA volume of solid of revolution x^3 - 0.3x^2 from 0.1 to 0.28

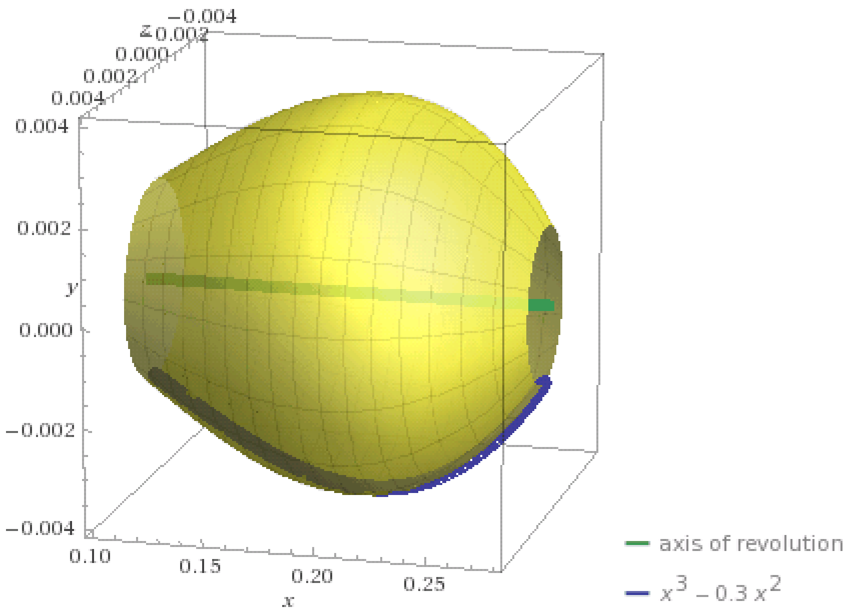
Result:

$$\int_{0.1}^{0.28} \pi\,(-0.3+x)^2\,x^4\,dx = 6.19195\times 10^{-6}$$

Surface area of solid:

0.00368816

Plot:



(axes not equally scaled)

WA surface area of solid of revolution x^3 - 0.3x^2 from 0.1 to 0.28

Result:

$$\int_{0.1}^{0.28} 2\,\pi\,\sqrt{1+9.\,(-0.2+x)^2\,x^2}\;\left|(-0.3+x)\,x^2\right|\,dx = 0.00366787$$

Volume of solid:

$$\int_{0.1}^{0.28} \pi\,(-0.3+x)^2\,x^4\,dx = 6.19195\times 10^{-6}$$

WA definite integral of 2piabs(x^3 - 0.3x^2)[1 + (3x^2 - 0.6x)^2]^(1/2) from 0.1 to 0.28

Definite integral:

$$\int_{0.1}^{0.28} 2\pi\left(\left|x^3 - 0.3x^2\right|\sqrt{1+\left(3x^2-0.6x\right)^2}\right)dx = 0.00366787$$

$|z|$ is the absolute value of z

Area at x=0.1: $\pi r^2 = \pi(f(x))^2 = \pi((0.1)^3 - 0.3(0.1)^2)^2 = 1.2566\text{x}10^{-5}$

Area at x=0.28: $\pi((0.28)^3 - 0.3(0.28)^2)^2 = 7.723995\text{x}10^{-6}$

$0.00368816 - (1.2566\text{x}10^{-5} + 7.723995\text{x}10^{-6}) = 0.00366787$

A6. WA volume of solid of revolution sin^2(x) - sin(3x) from 2 to 4

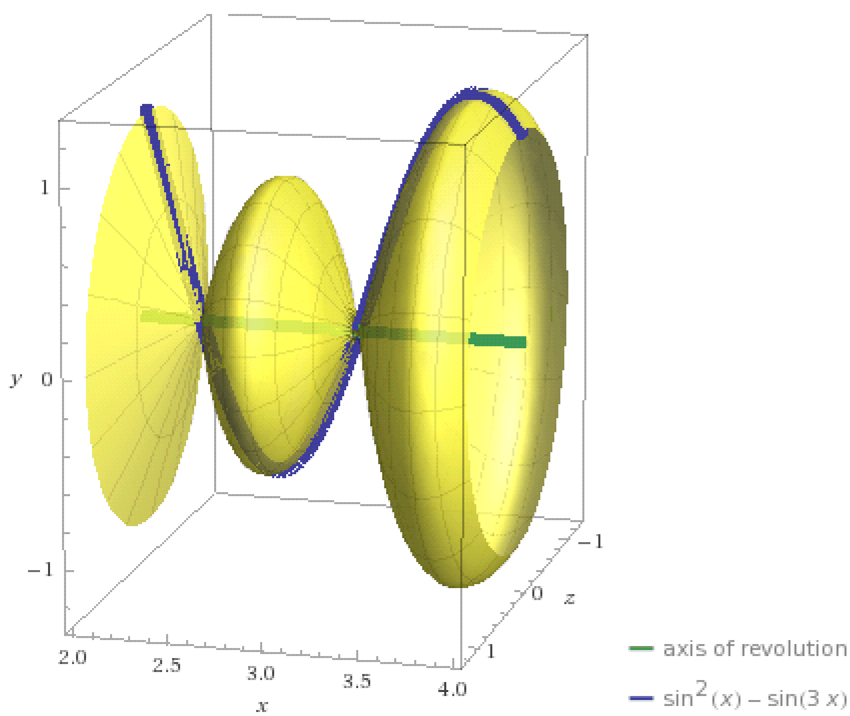
Result:

$$\int_2^4 \pi\left(\sin^2(x) - \sin(3x)\right)^2 dx = 3.95701$$

Surface area of solid:

25.4841

Plot:



WA surface area of solid of revolution sin^2(x) - sin(3x) from 2 to 4

Result: More digits

$$\int_2^4 2\pi \left|\sin^2(x) - \sin(3x)\right| \sqrt{1 + (3\cos(3x) - 2\cos(x)\sin(x))^2}\, dx \approx 17.773462349529\ldots$$

$|z|$ is the absolute value of z

Volume of solid:

$$\int_2^4 \pi \left(\sin^2(x) - \sin(3x)\right)^2 dx = 3.95701$$

WA definite integral of 2piabs(sin^2(x) - sin(3x))[1 + (sin(2x)-3cos(3x))^2]^(1/2) from 2 to 4

Definite integral:

$$\int_2^4 2\pi \left|\sin^2(x) - \sin(3x)\right| \sqrt{1 + (-3\cos(3x) + \sin(2x))^2}\, dx = 17.7735$$

$|z|$ is the absolute value of z

Area at x = 2: $\pi r^2 = \pi(f(x))^{2} = \pi(\sin^2(2) - \sin(3(2)))^2 = 3.84456$

Area at x = 4: $\pi(\sin^2(4) - \sin(3(4)))^2 = 3.86604$

25.4841 – (3.84456 + 3.86604) = 17.7735

A7. WA volume of solid of revolution (x^2 + 0.3cos(8x)) from 0.6 to 2.7

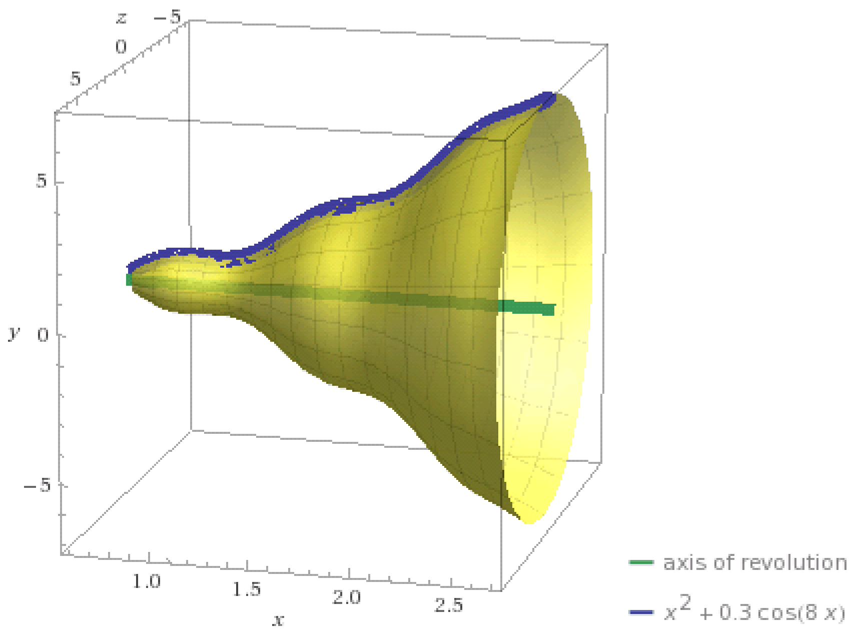
Result:

$$\int_{0.6}^{2.7} \pi \left(x^2 + 0.3 \cos(8\,x)\right)^2 \, dx = 90.9793$$

Surface area of solid:

316.793

Plot:



(axes not equally scaled)

WA surface area of solid of revolution (x^2 + 0.3cos(8x)) from 0.6 to 2.7

Result: More digits

$$\int_{0.6}^{2.7} 2\,\pi \left|x^2 + 0.3 \cos(8\,x)\right| \sqrt{1 + (2\,x - 2.4 \sin(8\,x))^2} \, dx \approx 161.829\ldots$$

$|z|$ is the absolute value of z

Volume of solid:

$$\int_{0.6}^{2.7} \pi \left(x^2 + 0.3 \cos(8\,x)\right)^2 \, dx = 90.9793$$

$(\, z \,)$ $\quad (\, \sin(\theta) \left(x_0^2 + 0.3 \cos(8\,x_0)\right)$

WA definite integral of 2piabs((x^2 + 0.3cos(8x)))[1 + (2x-2.4sin(8x))^2]^(1/2) from 0.6 to 2.7

Definite integral:

$$\int_{0.6}^{2.7} 2\pi\left(\left|x^2 + 0.3\cos(8x)\right|\sqrt{1 + (2x - 2.4\sin(8x))^2}\right) dx = 161.829$$

$|z|$ is the absolute value of z

Area at x=0.6: $\pi r^2 = \pi(f(x))^2 = \pi((0.6)^2 + 0.3\cos(8(0.6)))^2 = 0.4687$

Area at x = 2.7: $\pi((2.7)^2 + 0.3\cos(8(2.7)))^2 = 154.495$

316.793 – (0.469 + 154.495) = 161.829

***__**Note:**__ WA is sometimes very picky about the way formulas are entered. If I input:

calculate pi((0.6)^2 + 0.3cos(8*0.6))^2

or

calculate pi((0.6)^2 + 0.3cos(8(0.6)))^2

I get 1.36412, which is wrong, but if I input:

calculate pi((0.6)^2 + 0.3cos(4.8))^2

I get 0.4687, which is correct.

This graphic demonstrates what is happening:

WA calculate pi((0.6)^2 + 0.3cos(8*0.6))^2

Input:

$$\pi\left(0.6^2 + 0.3\cos((8\times 0.6)°)\right)^2$$

For whatever reason, WA is interpreting 8x0.6 as degrees instead of radians. This does not occur when you put in the calculated value of 4.8.

WA calculate pi((0.6)^2 + 0.3cos(4.8))^2

Input:

$$\pi\left(0.6^2 + 0.3\cos(4.8)\right)^2$$

When using WA to calculate values for formulas with a setup such as cos(8x) as in the formula above, calculate 8x first, then use that value (i.e. cos(4.8)) instead of inputting cos(8(0.6)) or cos(8*0.6) to avoid confusing WA. **Takeaway lesson: Always look at how WA interprets your inputs!**

A8. WA volume of solid of revolution (x^0.5 + 0.2sin(3x)) from 0 to 3

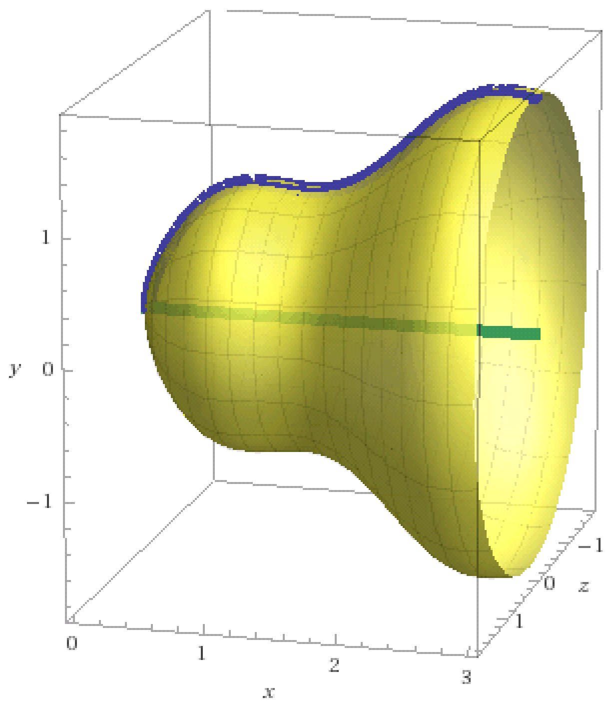
Result:

$$\int_0^3 \pi \left(\sqrt{x} + 0.2 \sin(3 x)\right)^2 d x = 15.1646$$

Surface area of solid:

36.8949

Plot:



WA surface area of solid of revolution (x^0.5 + 0.2sin(3x)) from 0 to 3

Result: More digits

$$\int_0^3 2 \pi \left|\sqrt{x} + 0.2 \sin(3 x)\right| \sqrt{1 + \left(\frac{1}{2\sqrt{x}} + 0.6 \cos(3 x)\right)^2}\, d x \approx 26.551734299\ldots$$

$|z|$ is the absolute value of z

Volume of solid:

$$\int_0^3 \pi \left(\sqrt{x} + 0.2 \sin(3 x)\right)^2 d x = 15.1646$$

WA definite integral of 2piabs(x^0.5 + 0.2sin(3x))[1 + (0.5x^(-0.5)-0.6cos(3x))^2]^(1/2) from 0 to 3

Definite integral:

$$\int_0^3 2\pi \left|\sqrt{x} + \frac{1}{5}\sin(3x)\right| \sqrt{1+\left(\frac{1}{2\sqrt{x}} - \frac{3}{5}\cos(3x)\right)^2}\, dx = 26.3029$$

$|z|$ is the absolute value of z

Area at x=0: $\pi r^2 = \pi(f(x))^2 = \pi((0)^{0.5} + 0.2\sin(3(0))^2 = 0$

Area at x=3: $\pi((3)^{0.5} + 0.2\sin(3(3))^2 = 10.3431$

36.8949 – (0 + 10.3431) = 26.5518

A9. WA solve 4x^2 - 36 = 0

x = -3, 3

WA volume of solid of revolution (4x^2 - 36) from -3 to 3

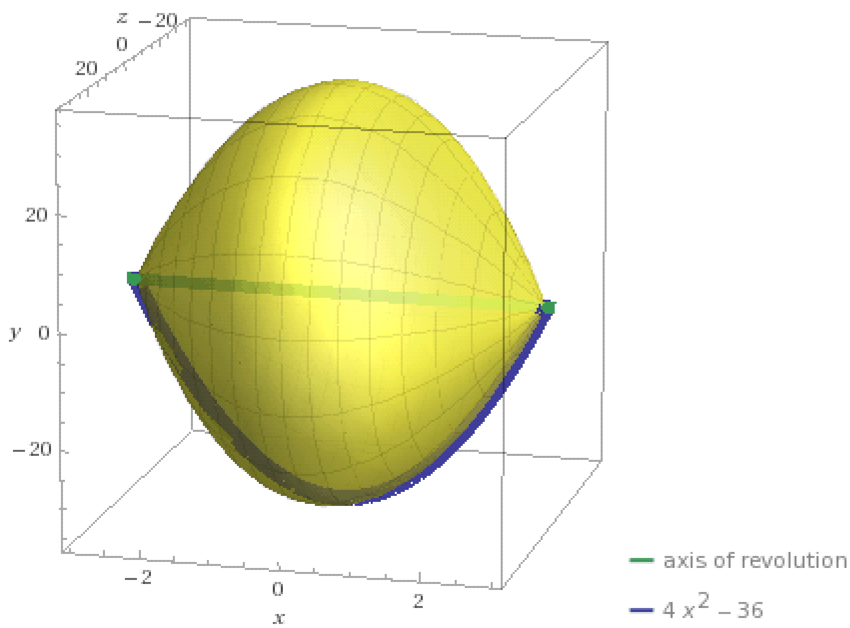
Result:

$$\int_{-3}^{3} 16\,\pi\,(-9+x^2)^2\,dx = \frac{20\,736\,\pi}{5} \approx 13\,028.8$$

Surface area of solid:

8252.51

Plot:



(axes not equally scaled)

WA surface area of solid of revolution (4x^2 - 36) from -3 to 3

Result:

$$\int_{-3}^{3} 8\,\pi\,\sqrt{1+64\,x^2}\,\left|-9+x^2\right|\,dx = 8252.51$$

$|z|$ is the absolute value of z

Volume of solid:

More digits

$$\int_{-3}^{3} 16\,\pi\,(-9+x^2)^2\,dx = \frac{20\,736\,\pi}{5} \approx 13\,028.8$$

A10. WA solve 0.5sin(6x) = 0

x = 0, $\pi/6$

WA volume of solid of revolution (0.5sin(6x)) from 0 to pi/6

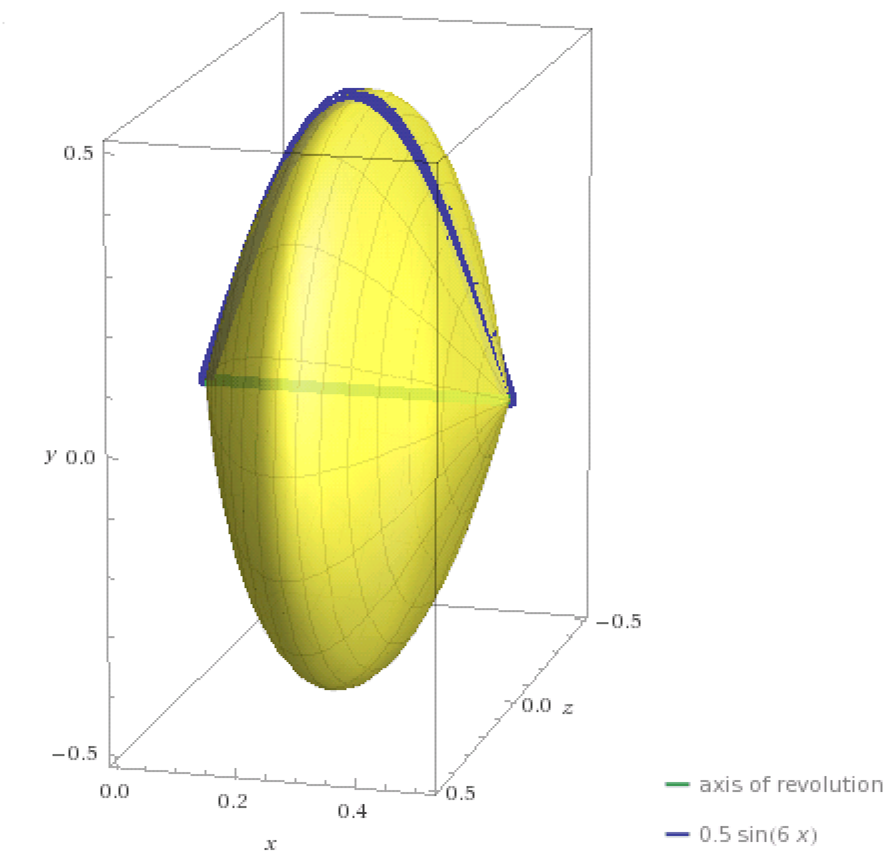
Result:

$$\int_0^{\frac{\pi}{6}} 0.785398 \sin^2(6\,x)\,dx = 0.205617$$

Surface area of solid:

1.97314

Plot:



WA surface area of solid of revolution (0.5sin(6x)) from 0 to pi/6

Result:

$$\int_0^{\frac{\pi}{6}} 3.14159\, |\sin(6\,x)| \sqrt{1 + 9.\cos^2(6\,x)}\; dx = 1.97314$$

$|z|$ is the absolute value of z

Volume of solid:

$$\int_0^{\frac{\pi}{6}} 0.785398 \sin^2(6\,x)\, dx = 0.205617$$

For Q11 – Q18, use the "surface of solid of revolution of region" method to find the surface area of f(x) rotated about the x-axis.

A11. f(x) = x from x = 0 to pi

WA surface of area of revolution of x about x axis from 0 to pi

Input interpretation:

surface area of surface of revolution	$y = x$	$x = 0$ to π	about the x-axis

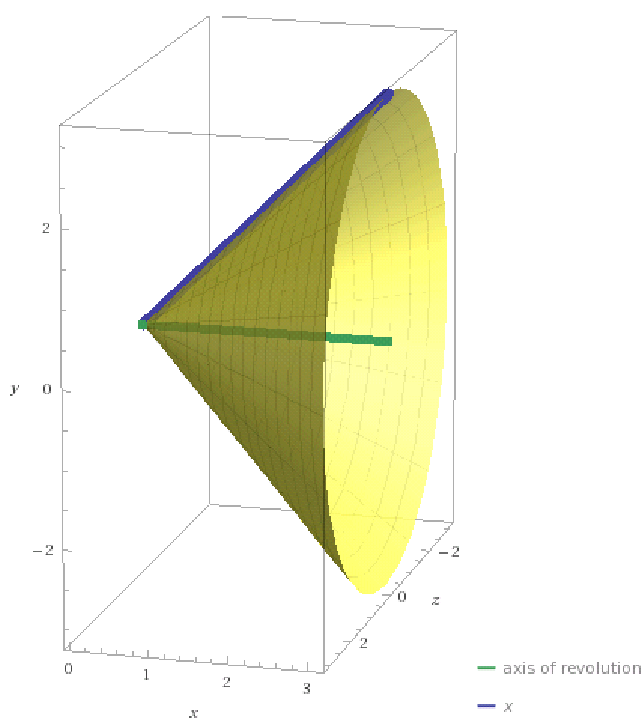
Result:

More digits

$$\int_0^{\pi} 2\sqrt{2}\,\pi\,|x|\, dx = \sqrt{2}\,\pi^3 \approx 43.8495$$

$|z|$ is the absolute value of z

Plot:



A12. f(x) = x^2 from x = 1 to 4

WA surface of area of revolution of x^2 about x axis from 1 to 4

Input interpretation:

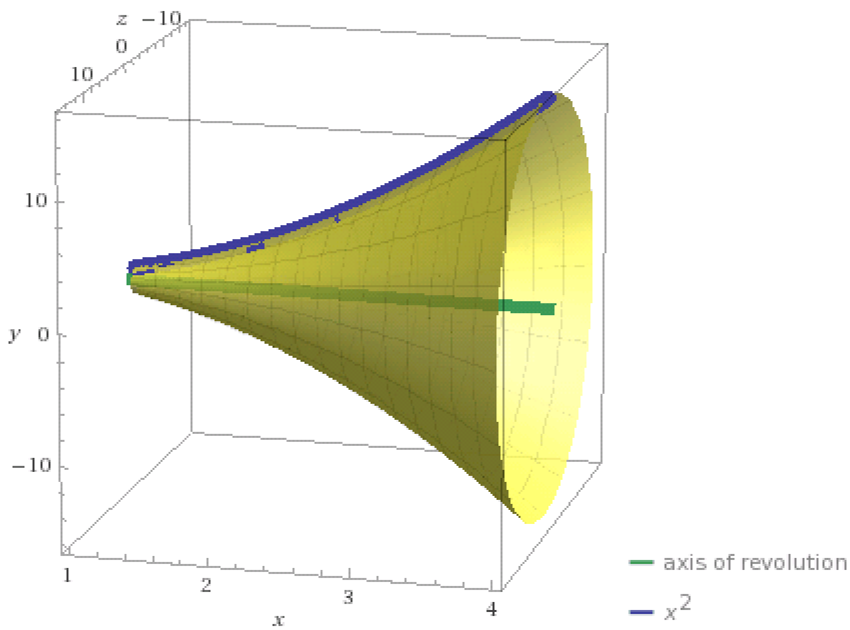
surface area of surface of revolution	$y = x^2$	$x = 1$ to 4	about the x-axis

Result:

$$\int_1^4 2\,\pi \sqrt{1+4\,x^2}\;|x|^2\,dx = 812.756$$

$|z|$ is the absolute value of z

Plot:



(axes not equally scaled)

A13. f(x) = e^x from x = 0 to 2

WA surface of area of revolution of e^x about x axis from 0 to 2

Input interpretation:

surface area of surface of revolution	$y = e^x$	$x = 0$ to 2	about the x-axis

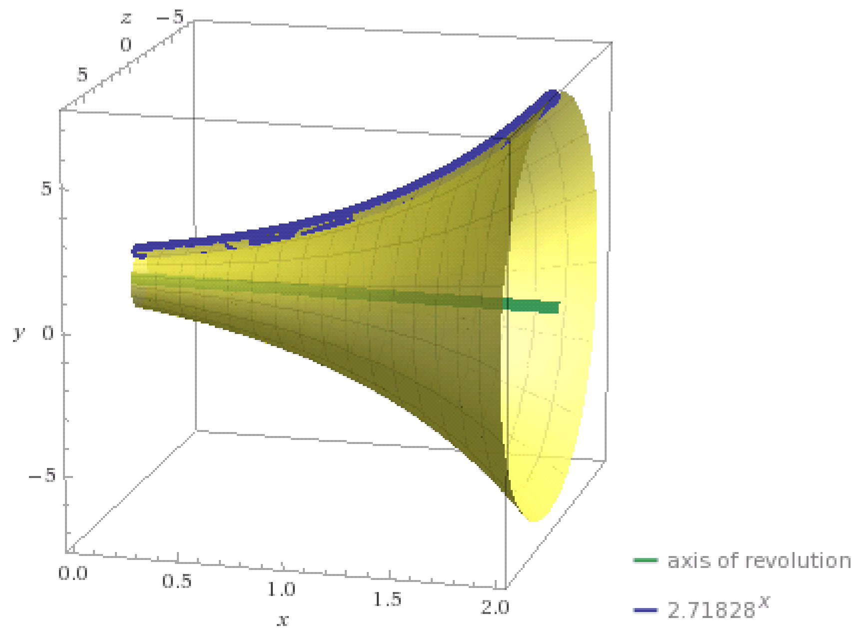
Result:

More digits

$$\int_0^2 2\,e^{\mathrm{Re}(x)}\sqrt{1+e^{2\,x}}\;\pi\,dx \approx 174.352075\ldots$$

$\mathrm{Re}(z)$ is the real part of z

Plot:



(axes not equally scaled)

A14. $f(x) = x^{1/2}$ from x = 0 to 1

WA surface of area of revolution of x^1/2 about x axis from 0 to 1

Input interpretation:

surface area of surface of revolution	$y = \sqrt{x}$	$x = 0$ to 1	about the x-axis

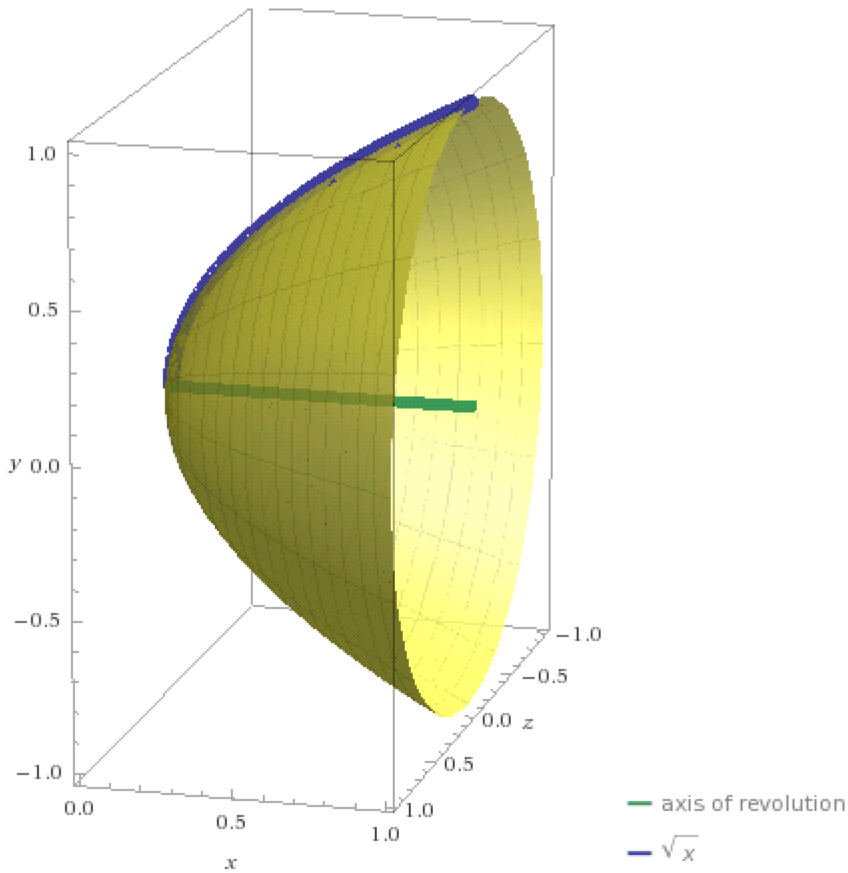
Result:

More digits

$$\int_0^1 2\pi \sqrt{1 + \frac{1}{4x}} \sqrt{|x|}\, dx = \frac{1}{6}\left(5\sqrt{5} - 1\right)\pi \approx 5.33041$$

$|z|$ is the absolute value of z

Plot:



A15. f(x) = sin(x) + 3 from x = 0 to 2pi

WA surface of area of revolution of (sin(x) + 3) about x axis from 0 to 2pi

Input interpretation:

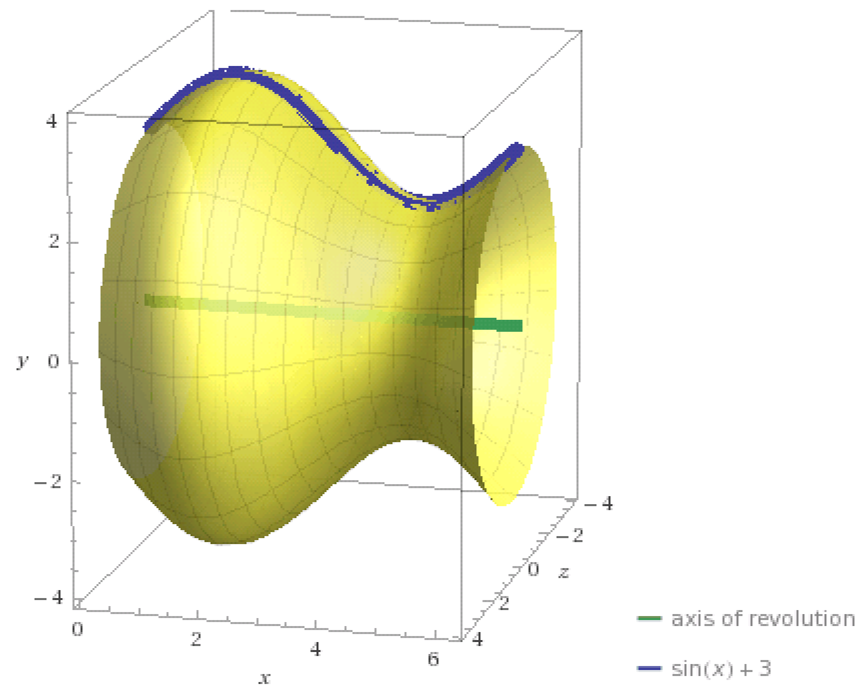
surface area of surface of revolution	$y = \sin(x) + 3$	$x = 0$ to 2π	about the x-axis

Result:

$$\int_0^{2\pi} 2\pi\,|3 + \sin(x)|\sqrt{1 + \cos^2(x)}\;dx = 144.018$$

$|z|$ is the absolute value of z

Plot:



A16. $f(x) = 3x + x^{1/2}$ from x = 0 to 3

WA surface of area of revolution of (3x + x^1/2) about x axis from 0 to 3

Input interpretation:

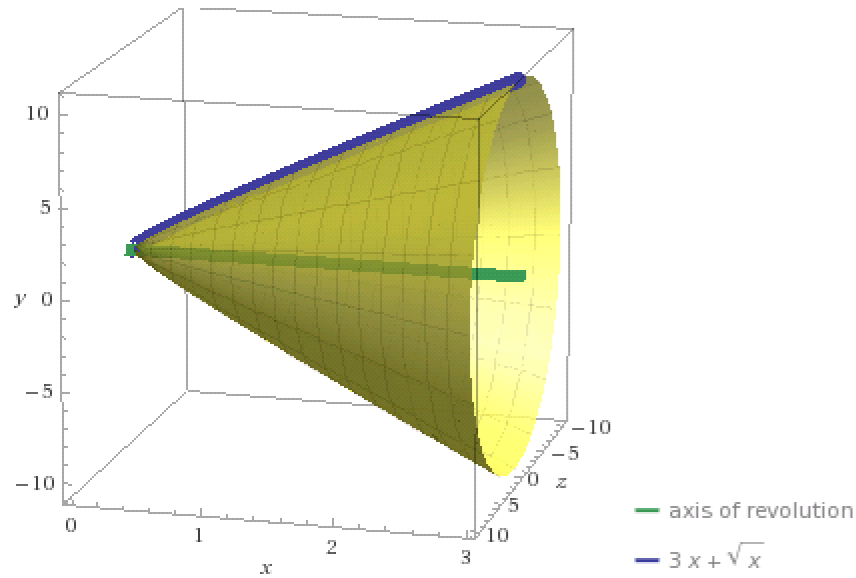
surface area of surface of revolution	$y = 3x + \sqrt{x}$	$x = 0$ to 3	about the x-axis

Result:

$$\int_0^3 2\pi \sqrt{1 + \left(3 + \frac{1}{2\sqrt{x}}\right)^2}\, |\sqrt{x} + 3x|\, dx = 377.237$$

$|z|$ is the absolute value of z

Plot:



(axes not equally scaled)

A17. f(x) = 2cos(3x) + 8 from x = 0 4pi

WA surface of area of revolution of (2cos(3x) + 8) about x axis from 0 to 4pi

Input interpretation:

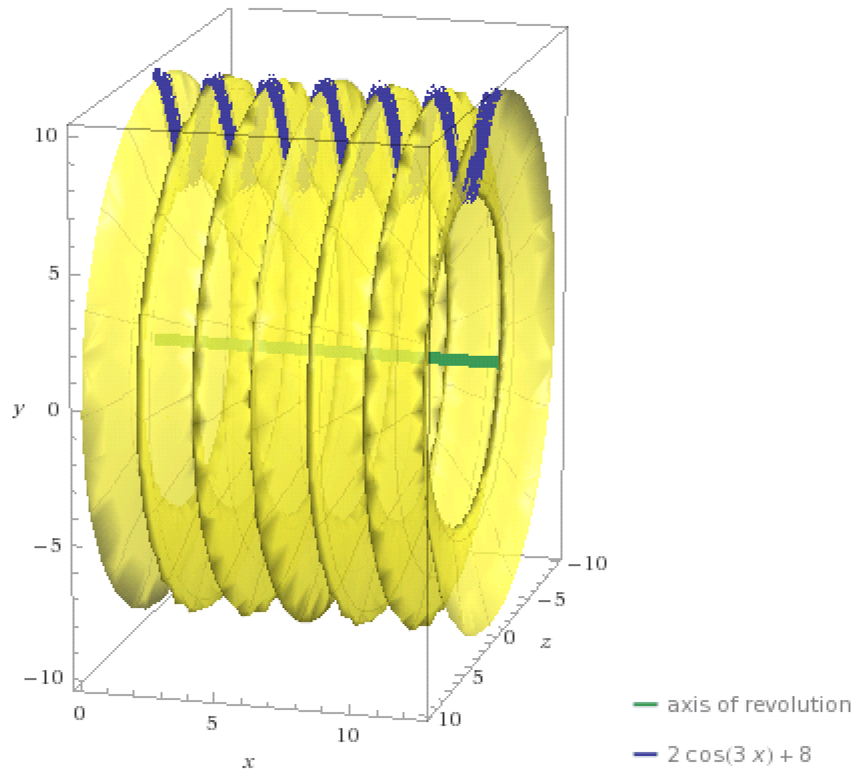
surface area of surface of revolution	$y = 2\cos(3x) + 8$	$x = 0$ to 4π	about the x-axis

Result:

$$\int_0^{4\pi} 4\pi\,|4 + \cos(3x)|\sqrt{1 + 36\sin^2(3x)}\,dx = 2535.72$$

$|z|$ is the absolute value of z

Plot:



A18. f(x) = $2x^2$ + 1 from x = 0 to 2

WA surface of area of revolution of (2x^2 + 1) about x axis from 0 to 2

Input interpretation:

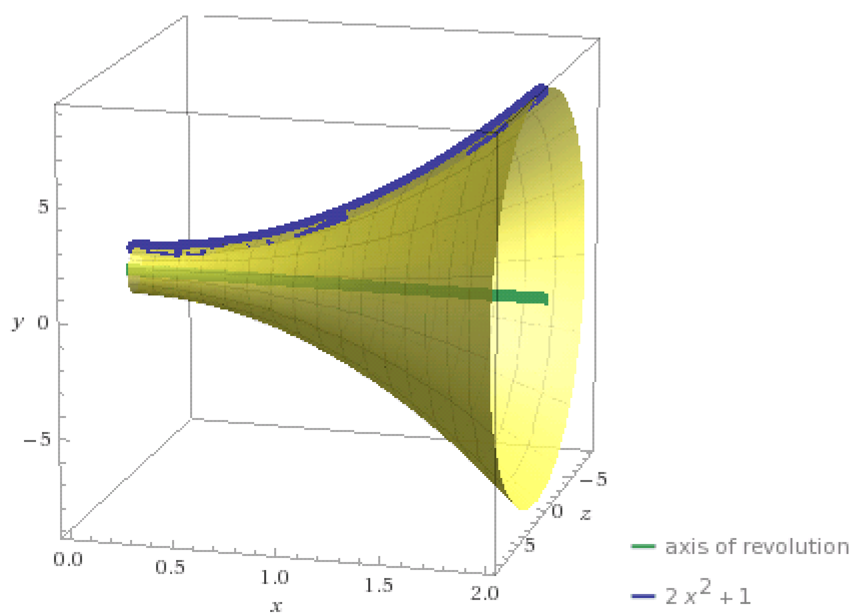
surface area of surface of revolution	$y = 2x^2 + 1$	$x = 0$ to 2	about the x-axis

Result:

$$\int_0^2 2\pi\sqrt{1+16x^2}\,|1+2x^2|\,dx = 256.979$$

$|z|$ is the absolute value of z

Plot:



(axes not equally scaled)

A19. Find the formula WA provided in the answer to the previous question (Q18). Which part of that formula for the surface area about the x-axis corresponds to the circumference of a disk?

Result:

$$\int_0^2 2\pi\sqrt{1+16x^2}\,\left|1+2x^2\right|\,dx = 256.979$$

Which part corresponds to the radius of that disk?

Result:

$$\int_0^2 2\pi\sqrt{1+16x^2}\,\left|1+2x^2\right|\,dx = 256.979$$

A20. What would you change in our surface area formula if you wanted to find the surface area of a function rotated only halfway around the x-axis?

Answer: You would change the 2pi to a pi, causing the arc to make only a semicircle.

Result:

$$\int_0^2 2\,\pi \sqrt{1+16\,x^2}\;\left|1+2\,x^2\right|\,dx = 256.979$$

T5 C30 Wolfram Alpha - Modern Tool

As you know, Wolfram Alpha, WA, is a powerful tool that has revolutionized how STEM professionals can solve calculus problems, and much more too.

As a STEM student you should master the use of this tool. This treatment of calculus has been designed to send you down the path of this mastery.

The more you use WA the better you will understand it and master it.

WA is a very powerful tool with many features and functions. You have now been exposed to some of them. However, you will probably never learn them all or master them all. I certainly have not!

At some point in time you may want to learn and master WA's "parent", Mathematica. This powerful tool will let you write "programs" whereby you can perform all sorts of very sophisticated things. You will be able to construct models and to analyze complex data sets and situations.

One good way to do this today is to utilize the Raspberry Pi which comes loaded with Mathematica. It is the first computer to include Mathematica since Steve Jobs, NEXT computer back in the early days of Mathematica.

In Tier 5 you have learned to use many Wolfram Alpha commands to solve problems using calculus concepts.

This is a list of those commands.

You may find explanations and examples in the appropriate lessons.

These are in the order they were introduced in Tier 5.

I. Graphing a function, f(x), Lessons C2 and C2a

WA1 Plot f(x)

WA2 Plot f(x) from a to b

WA3 Roots f(x)

WA4 Stationary Points f(x)

WA5 Inflection Points f(x)

WA6 Asymptotes f(x)

Plus three more covered later in Part 1

WA8 Plot f(x,y) = 0 from x = a to b (T5 C8)

WA9 Solve for y when x = a, f(x,y) = 0

WA10 Tangent line when x = a, f(x,y) = 0

You will note that no calculus is needed to understand and apply these Graphing commands.

The Graph of a function is probably the best way to understand its behavior.

As you can see, WA probably is easier to use and better than the classical graphing calculators. It certainly will then do things no calculator can do as we learned.

II. Part 1 Differentiation Lessons C3 thru C12

WA7 Derivative f(x)

WA8 Plot f(x,y) = 0 from x = a to b (T5 C8)

WA9 Solve for y when x = a, f(x,y) = 0

WA10 Tangent line when x = a, f(x,y) = 0

WA11 Solve for x, f(x,a)

WA12 Derivative f(x,y) = 0

WA13 g(x,y) at x = a and y = b

WA14 derivative inverse f(x) (T5 C10)

WA15 series f(x) (T5 C11)

WA16 series f(x) at x = a

WA17 series f(x) at x = a order n

III. Part 2 Integration Lessons C13 thru C24

WA18 antiderivative f(x)

WA18a indefinite integral f(x)

WA18b integrate f(x)

WA19 integrate f(x) from a to b

WA20 integrate absolute value f(x) from a to b

WA21 area f(x) from a to b

WA22 arc length f(x) from a to b

WA23 volume of revolution f(x) from x= a to b

WA24 volume of solid of revolution of region between f(x) and g(x) from a to b about the x axis.

WA25 surface area of solid of revolution f(x) from x = a to b about the x axis

WA26 Parametric Plot (f(t),g(t))

WA27 Parametric Plot (f(t),g(t)) from t = a to b

WA28 Arc Length (f(t),g(t)) from t = a to b

WA29

y – g(t) = ((Derivative g(t)) / (Derivative f(t)))(x-f(t)), t=a

This is the tangent line to (f)t),g(t)) at t = a

WA30 Integrate (Derivative f(t))*g(t)) from t = a to b

This is area under the Parametrically defined curve (f(t),g(t) from t = a to b, assuming no fold back.

WA31 Keep in mind the integrate from a to b command also works when there is a vertical asymptote c between a and b when this improper integral is finite.

WA32 Keep in mind the integrate command also works for a to infinity or a to – infinity when the improper integral exists.

In T5 C29 I left you with an Exercise to play with.

Exercise: Play with this. I will discuss it in C30

Find the region enclosed by x^2 and 3sin(x) and rotate it about the x axis and find its volume and surface area.

This will give you some good experience with WA and thinking about how to use it.

First, let's just plot it and find the points of intersection.

WA1 Plot x^2,3sin(x)

WA2 Solve x^2 = 3sin(x)

Note: We also get the plot here too.

WA3 Volume of solid of revolution of 3sin(x)-x^2 about the x axis Answer: 6.74419

WA4 Volume of solid of revolution of x^2 about the x axis from x=0 to 1.722 Answer: 9.51362

WA5 Volume of solid of revolution of 3sin(x) about the x axis from x=0 to 1.722 Answer 26.4494

So, the answer is 26.4494 – 9.5638 = 16.8856

WA6 Plot   x^2 - 3sin(x)

What about surface area?

We could use the surface are of the two volumes, but what must we do?

Subtract the two discs at the ends

Pix1.722^4 = 27.624

57.4095 – 27.624 = 29.786

66.0825 – 27.624 = 38.4585

WA7 Surface area of solid of revolution of   x^2 about the x axis from x=0 to 1.722

WA8 Surface area of solid of revolution of   3sin(x) about the x axis from x=0 to 1.722

Final thoughts and advice.

Some schools still insist on your learning the classical manual calculus techniques and tools. In particular, the techniques of manual integration by finding an antiderivative to then applying the FTC.

Personally, I consider this a waste of time for STEM students and would not recommend any such school to a student.

I hope your school will allow you to skip the traditional calculus course. You will not use these classical manual tools in your work any more than you would use a slide rule or trig or log tables. I'm pretty sure the science and engineering departments would be fine with this. The problem might be with the math departments and the school's graduation requirements.