Tier 5 Calculus Lesson 1 Notes:
Approach to learning Calculus

Our approach to learning Calculus will be what is called "Heuristic."

From Wikipedia here is the definition of Heuristic.

Heuristic (/hjʉˈrɪstɨk/; Greek: "Ε□ρίσκω", **"find"** or **"discover"**) refers to experience-based techniques for problem solving, learning, and discovery that find a solution which is not guaranteed to be optimal, but good enough for a given set of goals. Where the exhaustive search is impractical, heuristic methods are used to speed up the process of finding a satisfactory solution via mental shortcuts to ease the cognitive load of making a decision.

We will use the modern tool Wolfram-Alpha, WA, to explore and learn the concepts of calculus, and their application in solving problems.

Many introductory calculus courses attempt to apply Rigor to the treatment and learning of calculus. We feel this is premature for the first exposure or pass through the learning of calculus.

From Wikipedia here is the definition of Rigour, or Rigor.

Mathematical rigour can refer both to rigorous methods of mathematical proof and to rigorous methods of mathematical practice (thus relating to other interpretations of rigour).

Mathematical rigour is often cited as a kind of gold standard for mathematical proof. It has a history traced back to Greek mathematics, in the work of Euclid. This refers to the axiomatic method.

During the 19th century, the term 'rigorous' began to be used to describe decreasing levels of abstraction when dealing with calculus which eventually became known as analysis. The works of Cauchy added rigour to the older works of Euler and Gauss. The works of Riemann added rigour to the works of Cauchy. The works of Weierstrass added rigour to the works of Riemann, eventually culminating in the arithmetization of analysis. Starting in the 1870s, the term gradually came to be associated with Cantorian set theory. [And, the Zermelo-Frankel Set Theory, and ultimately Non-Standard Analysis]

Most mathematical arguments are presented as prototypes of formally rigorous proofs. The reason often cited for this is that completely rigorous proofs, which tend to be longer and more unwieldy, may obscure what is being demonstrated.

Steps which are obvious to a human mind may have fairly long formal derivations from the axioms. Under this argument, there is a trade-off between rigour and comprehension.

Historically, "infinitesimals" were used in a heuristic and intuitive approach to understanding the concepts of calculus and "proving" the various theorems and results.

Infinitesimals were used by all of the great mathematicians starting with Archimedes, and proceeding with Newton and Leibniz, the inventors of calculus, and Euler who essentially created modern calculus and differential equations and much more.

However, in the mid 1800's mathematicians decided that the heuristic and intuitive approach, especially to infinite series and other infinite processes, was inadequate and needed to be treated with rigor.

The Real and Complex Number systems were treated rigorously utilizing both a constructive and an axiomatic approach. Then calculus was treated with rigor also, using what has become known as the ε, δ method.

Truly modern rigor is only achieved in a modern analysis course, usually in graduate school. This involves a modern treatment of the number systems and what is called "set theory" and "measure theory".

In the mid 1800's mathematicians could not figure out any way to include the concept of infinitesimal numbers in a rigorous treatment of the number systems. So they <u>banned infinitesimals</u> from modern mathematics!

Engineers and some scientists continued to use them in their intuitive learning and heuristic problem solving. There was quite a "split" between applied mathematicians, engineers, scientists and theoretical mathematicians.

Calculus textbooks started to apply some "rigor" in the mid 20th Century. Unfortunately, this made learning calculus for the first time quite laborious and difficult. Mastering a concept rigorously, or even quasi-rigorously, is much more difficult than understanding the concept heuristically and intuitively.

Consequently, many students "failed" in their study of calculus and subsequently were alienated from mathematics and sometimes blocked from pursuing a STEM career. I witnessed this first hand in the mid 20th century, and find it is even worse today.

Ironically, in the 1960's mathematicians finally succeeded in including infinitesimals into a rigorous treatment of numbers called nonstandard analysis. Now the heuristic approach to calculus using infinitesimals could be made just a rigorous as the ε, δ approach.

I believe that the best approach to learning calculus for the first time is to use the heuristic and intuitive approach with infinitesimals. That is what we will do in this treatment of calculus. All of our infinitesimal arguments can be made completely rigorous by modern standards.

I also believe in using the modern tool of Wolfram-Alpha, WA, to both understand concepts and solve problems is the very best way to proceed in learning calculus, especially for STE students. Rigor for future mathematicians should be put off until a second pass through the subject.

The result is that a student can master the concepts of calculus and their applications utilizing WA in a much shorter span of time than the current standard curriculum approach to calculus.

Tier 5 Calculus Lesson 1 Overview: What is it?

Functions are the main way STEM professionals represent various things they are studying. Understanding the various things one needs to know about functions is one major Key to success in STEM.

"Calculus" is an important set of tools utilized in the analysis of and understanding of "Functions".

A function, f, is the relationship between two variables, which represent two entities.

y = f(x) indicates a relationship between the "independent variable x" and the "dependent variable y". In calculus, x and y are both real numbers.

The Graph of f, is the set of points in the x-y plane, (x,f(x)) where x is in the "domain" of the function. The quickest and easiest, way to understand the behavior of a function is to analyze its graph which reveals the function's behavior in a very intuitive geometric way, both locally and globally.

Calculus is the study of two things, "the rate of change" and "the accumulation of change" for any type of phenomenon that is modeled by some function.

'Rate of Change' is dealt with using the concept of "Derivative" or Differential Calculus

'Accumulation of Change' is dealt with using the concept of "Integral" or Integral Calculus.

Historically, in the 17th and 18th centuries mankind developed a set of tools which can be used to "solve calculus problems" that arise in STE, Science, Technology, and Engineering.

You may think of these tools as manual tools like those used in "old fashioned" carpentry or automotive mechanics or manufacturing.

Today, in the 21st century, we have automated tools like Wolfram Alpha, WA, that enable us to solve calculus problems, just as we have automatic tools for carpentry, or automotive mechanics or manufacturing.

While it might be interesting, or even instructive, to learn to use the old fashioned classical tools, it is imperative we master the new modern tools since these are what we STEM professionals will be using to solve calculus problems.

What is really important is to understand the "concepts" of calculus. That is what we will emphasize, along with learning the modern tool WA. Indeed, WA will also help us understand the concepts too.

We will include a limited treatment of the classical "hand tools" of calculus for those who might be interested in them, or those who might be tested on them by someone or some institution.

At this point in time (2015), calculus in most schools, both high school and university in the U.S., is still taught using the classical "hand tools". These are difficult to master, particularly the integral calculus tools, and cause many STE students to struggle with math.

Modern tools like WA make the learning and practice of calculus quite easy, just as a modern scientific calculator makes arithmetic orders of magnitude easier than the old fashioned classical manual techniques of calculation.

In Tier 5 we will present calculus primarily utilizing WA, and offer the classical approach as a supplement for those students interested in this approach for some reason. This classical approach might best be left for a second pass through the program.

As you might expect, calculus is easy to learn and master utilizing the modern tool WA.

Any 21st century STE student should be utilizing a modern tool like WA when learning a STE subject and solving STE problems.

Before we begin our study of calculus, we will begin by studying the graphs of functions, since we can use WA to create these graphs.

This will then make it much easier to understand the concepts of calculus, and the subsequent use of the modern tools to solve calculus problems that arise in all STEM subjects.

Unfortunately, we know of NO calculus textbook that approaches the subject in this modern 21st century way.

So, you will have to rely on these notes, and the notes you will be generating for yourself as you learn and use WA.

T5 C1 Approach to Learning Calculus – Exercises

Q1. What does ‘heuristic’ mean?

Q2. What is mathematical rigour?

Q3. What does ‘rigorous’ mean?

Q4. What is an axiom?

Q5. The ban of infinitesimals caused a split between those using a heuristic approach and those using a rigorous approach. In the 1960’s, mathematicians finally succeeded in including infinitesimals into a rigorous treatment of numbers using what?

Q6. What modern tool allows students to both understand calculus concepts and solve problems?

Q7. What is a function?

Q8. In $y = f(x)$, which variable is the dependent variable?

Q9. In $y = f(x)$, which variable is the independent variable?

Q10. What is calculus?

A1. ‘Heuristic’ refers to experience-based techniques for problem solving, learning, and discovery that find a solution which is not guaranteed to be optimal, but good enough for a given set of goals.

A2. **Mathematical rigour** can refer both to rigorous methods of mathematical proof and to rigorous methods of mathematical practice (thus relating to other interpretations of rigour).

A3. ‘Rigorous’ is using an “axiomatic” approach to prove theorems about a mathematical subject such as calculus.

A4. Axiom is a rule or a statement that is accepted as true without proof.

A5. Nonstandard analysis

A6. Wolfram Alpha

A7. The relationship between two variables

A8. y

A9. x

A10. Calculus is the study of two things, “the rate of change” and “the accumulation of change” for any type of phenomenon that is modeled by some function.

Tier 5 Calculus Lesson2 Notes:
Function Graph Terms Definitions

Function f : y = f(x), x ε D ≤ R, y ε R

D = Domain = Set of real numbers f(x) is defined for.

R = Real Numbers,

≤ means "is a subset of", ε means "is contained in"

Graph of f, Set of (x, f(x)) in plane, x ε D

Terms describing f, at (a, f(a)) for any number a ε D

Defined **f(a) is defined, a ε D**

Continuous **f(a) = Limit of f(x) as x → a**

Smooth **Has tangent line at (a, f(a))**

Increasing **Left to Right Graph is going up**

Decreasing **Left to Right Graph is going down**

Stationary Point **Tangent Line is horizontal, slope 0.**

Max or Maximum (Local) $f(a - h) < f(a) > f(a + h)$

Min or Minimum (Local) $f(a - h) > f(a) < f(a + h)$

Concave Up **Tangent line slope increasing**

Concave Down **Tangent line slope decreasing**

Inflection Point at (a, f(a)) **Concavity switching directions**

Vertical Asymptote (Usually a not in Domain)

$f(x) \to \infty$ **or** $f(x) \to -\infty$ **as** $x \to a$ **from left or right**

Asymptote g(x) when $x \to \infty$ **or** $x \to -\infty$

$f(x) \to g(x)$ **as** $x \to \infty$ **or** $x \to -\infty$

T5 C2 Graph Examples

Plot Absolute Value x

Plot x/(Absolute Value x)

Plot (x-1)^2 + 3

Plot (x-1)^3 + 3

Plot (x+2)/((x-1)(x+1)) from x=-3 to 2

You will also want to use commands:

Roots

Stationary Points

Inflection Points

Asymptotes

See examples in T5 C2a

T5 C2 Function Graph Terms Definitions – Exercises

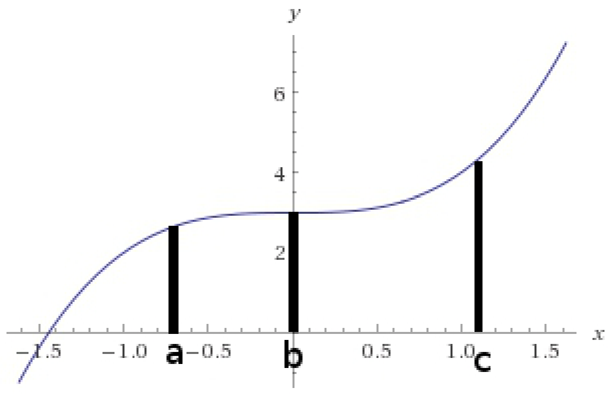


Figure 1

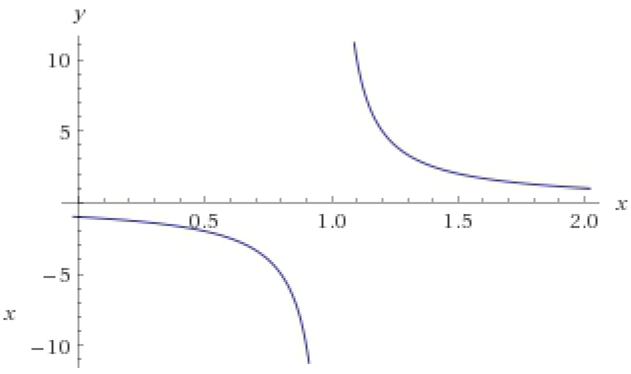


Figure 2

Q1. In Figure 1, is the function continuous?

Q2. In Figure 1, at **Point "a"** is the function concave up or concave down?

Q3. In Figure 1, at **Point "c"** is the function concave up or concave down?

Q4. In Figure 1, **Point "b"** is known as a(n)?

Q5. In Figure 2, is the function continuous?

Q6. In Figure 2, what occurs in the function at the value x=1?

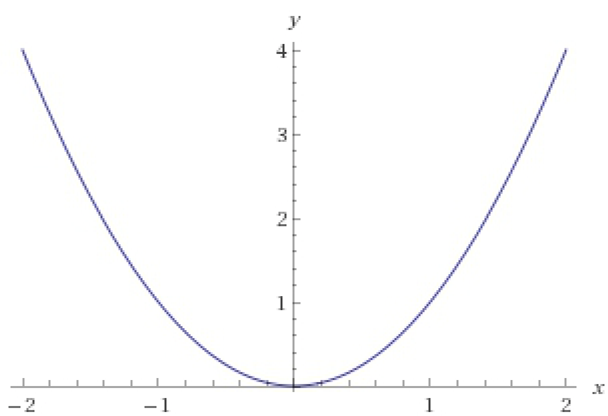


Figure 3

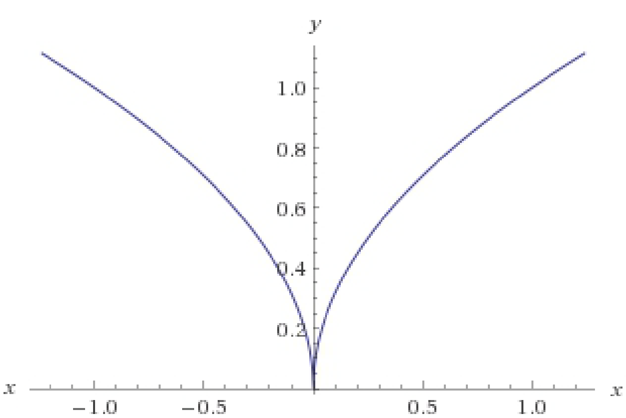


Figure 4

Q7. In Figure 3, the function at the value x=0 is a minimum or maximum?

Q8. In Figure 3, is the function at the value x=0 a stationary point?

Q9. In Figure 4, is the function continuous?

Q10. In Figure 4, is the function smooth?

A1. Yes

A2. Concave down

A3. Concave up

A4. Inflection point

A5. No

A6. Vertical asymptote

A7. Minimum

A8. Yes

A9. Yes

A10. No

Tier 5 Calculus Lesson 2A Notes: Graph Examples

NOTE: You should use Wolfram Alpha and enter each of these instructions and print them out so you can follow what I am doing in the video.

Pause the video each time and enter the example into WA and print it out and then try to understand the print out, and then continue the video.

You will learn "by doing", not just watching me.

I used a slightly condensed version of these notes in the video, but all the examples are the same.

Example 1

Plot ((x+1)^2*(x-2)^2)/((x-1)*(x+2)) from x=-5 to 5

Roots ((x+1)^2*(x-2)^2)/((x-1)*(x+2))

Stationary Points ((x+1)^2*(x-2)^2)/((x-1)*(x+2))

Inflection Points ((x+1)^2*(x-2)^2)/((x-1)*(x+2))

Asymptotes ((x+1)^2*(x-2)^2)/((x-1)*(x+2))

Example 2

Plot y=x^5 -3x^4 -6x^3 +15x^2 +4x -12

Roots x^5 -3x^4 -6x^3 +15x^2 +4x -12

Stationary Points x^5 -3x^4 -6x^3 +15x^2 +4x -12

Inflection Points x^5 -3x^4 -6x^3 +15x^2 +4x -12

Asymptotes x^5 -3x^4 -6x^3 +15x^2 +4x -12

Example 3

Plot (x^2sin(x))/ln(x^6)

Roots (x^2sin(x))/ln(x^6)

Stationary Points (x^2sin(x))/ln(x^6)

Stationary Points (x^2sin(x))/ln(x^6) from x = -30 t0 30

Inflection Points (x^2sin(x))/ln(x^6)

Inflection Points (x^2sin(x))/ln(x^6) from x = 2 to 20

Asymptotes (x^2sin(x))/ln(x^6)

NOW, do the exercises and learn WA by playing with it.

You will learn any STEM subject by playing with it and doing many exercises.

It will be of great benefit to you if you make up your own exercise too.

T5 C2A Graph Examples – Exercises

In Wolfram Alpha (WA), plot the following functions, as well as find all roots, stationary points, inflection points, and asymptotes. NOTE: Most of these examples here are much too difficult to do manually. This demonstrates the power of WA. However, even WA sometimes fails, and you must use your judgment. Always refer to the Plot or Graph to check WA's answers.

You can go to any calculus book and answer these questions for any of their problems.

Q1. $f(x) = 4x^5 - 6x^4 - 3x^3 + 8x^2 + 3x - 6$

Q2. $f(x) = \dfrac{(x-3)^2(-x+6)^3}{(x-6)^2(2x+4)}$

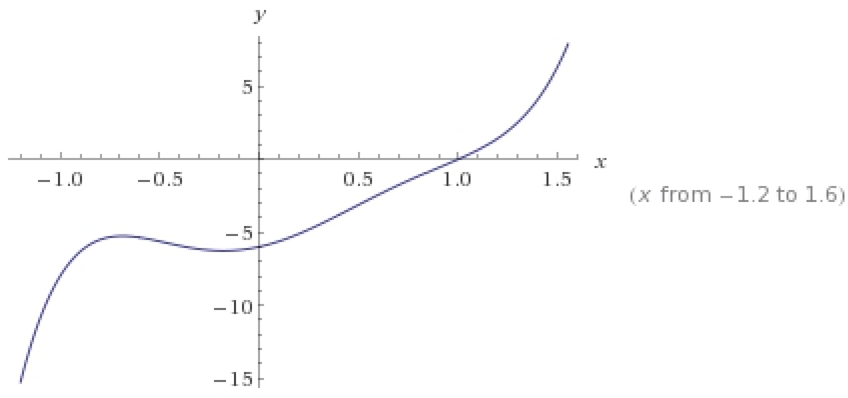
Q3. $f(x) = \dfrac{\sin(2x)}{\tan(3x)}$

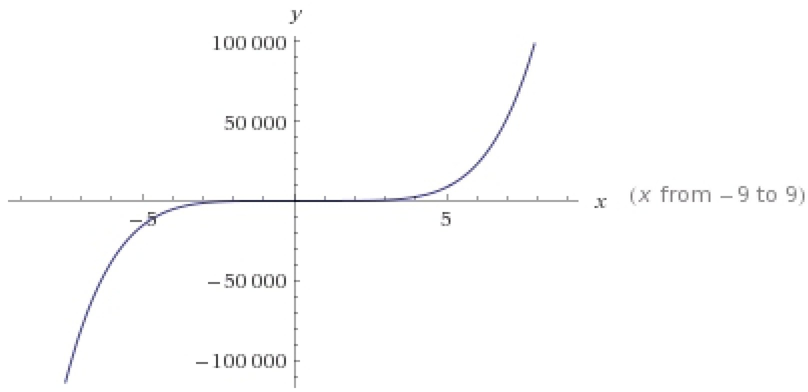
Q4. $f(x) = \dfrac{-(3x-2)((x+6)}{(x+3)}$

Q5. $f(x) = \sin(3x) - 4x^2 + 3x^3$

Q6. $f(x) = \dfrac{\sin(5x)}{3x^3+4x^2}$

A1. WA plot 4x^5 -6x^4 - 3x^3 +8x^2 +3x -6





WA roots 4x^5 -6x^4 - 3x^3 +8x^2 +3x -6

x = 1
x ~~ -0.857559-0.431517 i
x ~~ -0.857559+0.431517 i
x ~~ 1.10756-0.633158 i
x ~~ 1.10756+0.633158 i

Note: Only one real root at x = 1

WA stationary points 4x^5 -6x^4 - 3x^3 +8x^2 +3x -6

4 x^5-6 x^4-3 x^3+8 x^2+3 x-6~~-5.26118 at x~~-0.687089 (maximum)
4 x^5-6 x^4-3 x^3+8 x^2+3 x-6~~-6.27036 at x~~-0.179353 (minimum)

Note: Compare them with the graph to see if you believe them.

WA inflection points 4x^5 -6x^4 - 3x^3 +8x^2 +3x -6

4 x^5-6 x^4-3 x^3+8 x^2+3 x-6~~-5.7085 at x~~-0.472418
4 x^5-6 x^4-3 x^3+8 x^2+3 x-6~~-3.34795 at x~~0.468199
4 x^5-6 x^4-3 x^3+8 x^2+3 x-6~~-0.557443 at x~~0.904219

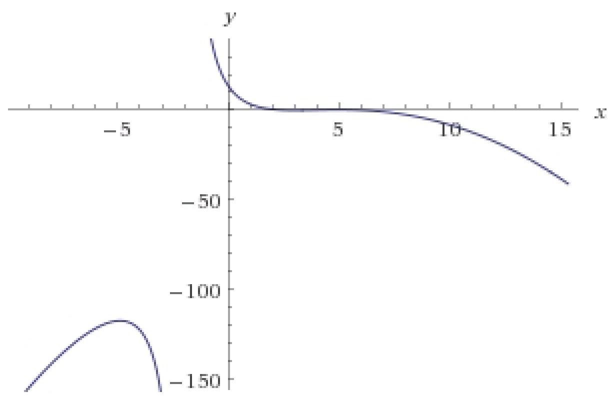
Note: Again compare to graph. See how they agree.

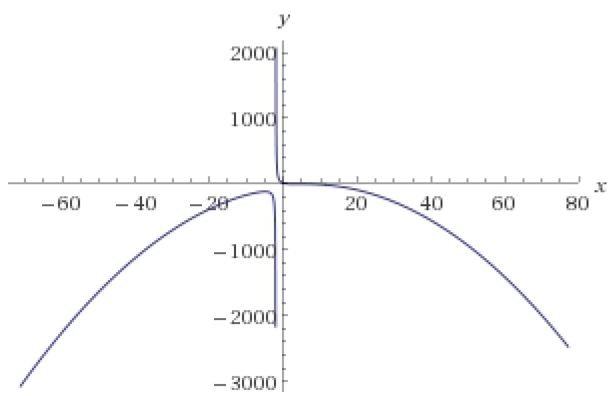
WA asymptotes 4x^5 -6x^4 - 3x^3 +8x^2 +3x -6

4 x^5-6 x^4-3 x^3+8 x^2+3 x-6 is asymptotic to 4 x^5-6 x^4-3 x^3+8 x^2+3 x-6

Note: It is asymptotic to itself. This will be true for all polynomials.

A2. WA plot ((x-3)^2(-x+6)^3)/((x-6)^2(2x+4))





WA roots ((x-3)^2(-x+6)^3)/((x-6)^2(2x+4))

x = 3

Note: This is clearly wrong. x = 6 is a root also just looking at the numerator. What happened?

simplify ((x-3)^2(-x+6)^3)/((x-6)^2(2x+4)) and you will get ((x-3)^2(-x+6))/(2x+4)

WA roots ((x-3)^2(-x+6))/(2x+4) will yield x = 3 and x = 6.

So Wolfram Alpha made a mistake when (x-6) was duplicated in both numerator and denominator.

You must always double check. I like to check the results against the graph. That is how I spotted this mistake, and then it was easy to correct.

WA is a great tool, but not infallible.

WA stationary points ((x-3)^2(-x+6)^3)/((x-6)^2(2x+4))

= 0 at x = 3 (minimum)
= -3/2 (39+16 sqrt(6)) at x = -2 sqrt(6) (maximum)
= -117/2+24 sqrt(6) at x = 2 sqrt(6) (maximum)

Looks good if you study the graph.

WA inflection points ((x-3)^2(-x+6)^3)/((x-6)^2(2x+4))

~~0.132319 at x~~3.84804

Looks good.

WA asymptotes ((x-3)^2(-x+6)^3)/((x-6)^2(2x+4))

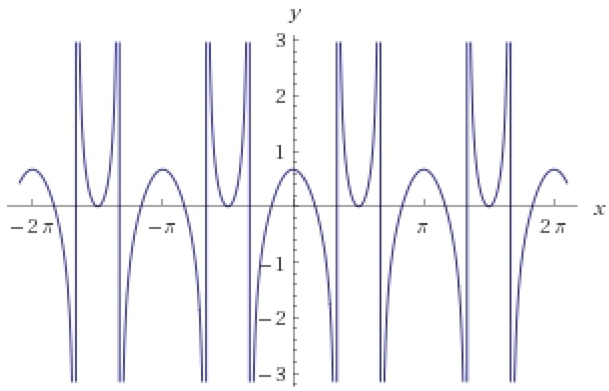
Vertical asymptotes
((6-x)^3 (x-3)^2)/((x-6)^2 (2 x+4))->±infinity as x->-2

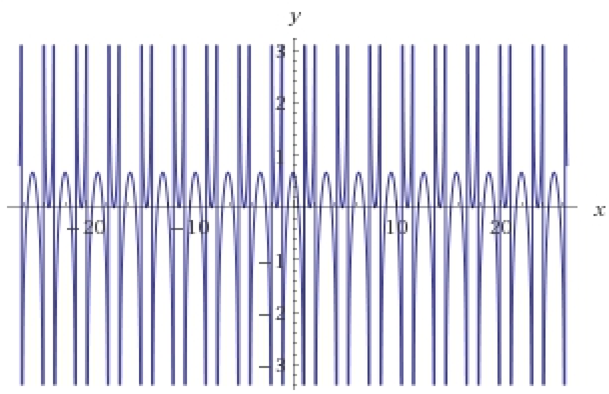
Note: Now WA recognizes that x = 6 is not an asymptote. This is due to fact that it (x-6) factor is not really there since it cancels.

Parabolic asymptotes
((x-3)^2 (-x+6)^3)/((x-6)^2 (2 x+4)) is asymptotic to -x^2/2+7 x-73/2

Looks good.

A3. WA plot sin(2x)/tan(3x)





WA roots sin(2x)/tan(3x)

x = 1/6 pi (2 n-1) and n element Z
x = pi (n-1/2) and n element Z

Looks good. Note there are infinitely many. So one for each integer n.

WA stationary points sin(2x)/tan(3x)

= 0 at x = 1/2 (4 n pi-pi) for integer n (minima)
= 0 at x = 1/2 (4 pi n+pi) for integer n (minima)

Looks good too

WA inflection points sin(2x)/tan(3x)

(no inflection points found)

This I believe from graph

WA asymptotes sin(2x)/tan(3x)

none

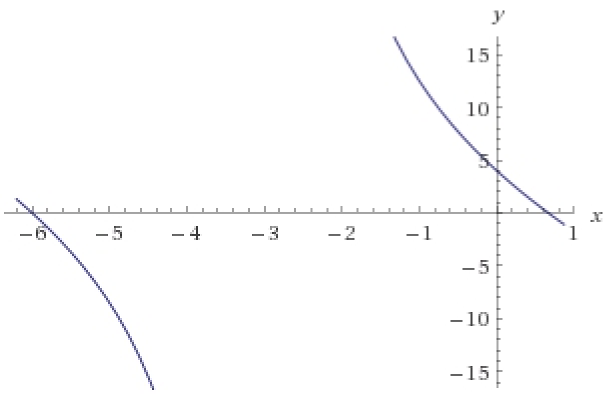
Oops. This is obviously not true from graph.

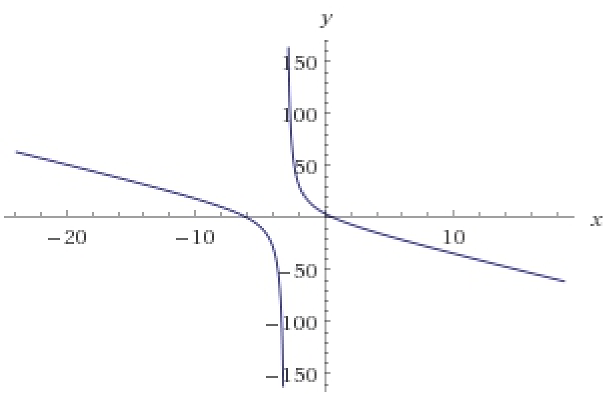
So find out when denominator is 0, and numerator not 0.

WA roots tan(3x)

x = nPi/3 for all integers n.

A4. WA plot -((3x-2)(x+6))/(x+3)





WA roots -((3x-2)(x+6))/(x+3)

x=-6
x=2/3

WA stationary points -((3x-2)(x+6))/(x+3)

(no stationary points found)

WA inflection points -((3x-2)(x+6))/(x+3)

(no inflection points found)

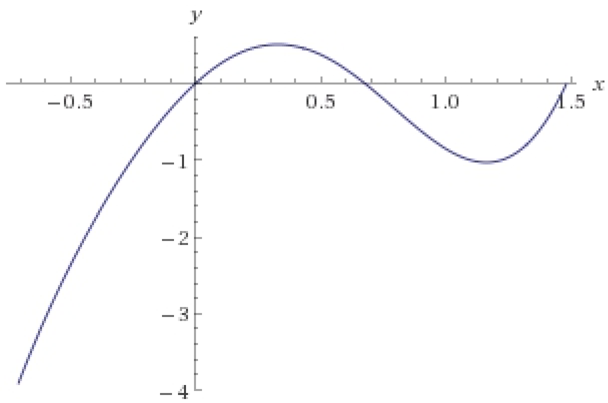
WA asymptotes -((3x-2)(x+6))/(x+3)

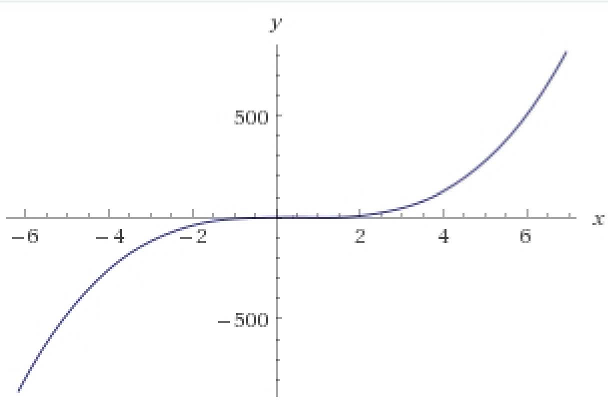
Vertical asymptotes
-((x+6) (3 x-2))/(x+3)->±infinity   as   x->-3

Oblique asymptotes
-((3 x-2) (x+6))/(x+3) is asymptotic to -3 x-7

All of these seem to check with the graph.

A5. WA plot sin(3x)-4x^2+3x^3





You might want to plot this function without the sin term first. Then with the sin term

You might want to plot this with sin(13x) from x = -2 to 2

Play with it.

WA roots sin(3x)-4x^2+3x^3

x = 0
x ~~ 0.674522
x ~~ 1.47991

WA stationary points sin(3x)-4x^2+3x^3

~~0.508247 at x~~0.328557
~~-1.0317 at x~~1.15997

WA inflection points sin(3x)-4x^2+3x^3

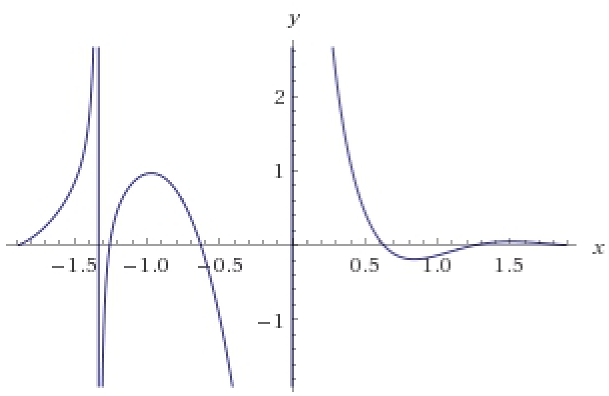
~~-0.32424 at x~~0.791484

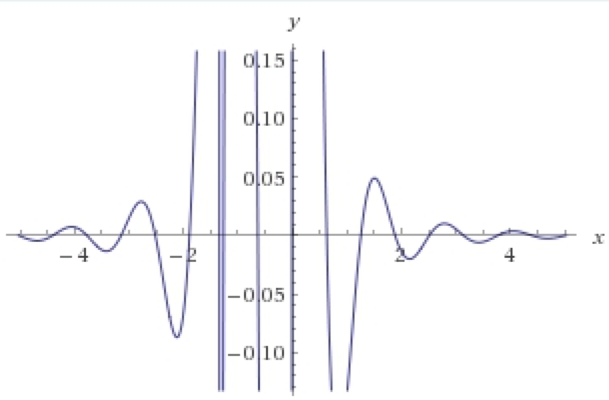
WA asymptotes sin(3x)-4x^2+3x^3

Polynomial asymptotes
sin(3 x)-4 x^2+3 x^3 is asymptotic to x^2 (3 x-4)

Note: it is just asymptotic to the polynomial without the sin term. This to be expected since the sin term can only add something between -1 and 1

A6. WA plot (sin(5x))/(3x^3+4x^2)





WA roots (sin(5x))/(3x^3+4x^2)

x = (2 pi n)/5 and n (3 pi n+10)!=0 and n element Z
x = 1/5 pi (2 n+1) and (2 n+1) (pi (6 n+3)+20)!=0 and n element Z

WA stationary points (sin(5x))/(3x^3+4x^2)

(no stationary points found)

Clearly, not true. So what to do?

Restrict domain, from x = -10 to 10

WA stationary points (sin(5x))/(3x^3+4x^2) from x = -10 to 10

Now there are many of them.

Results:

$$\frac{\sin(5x)}{3x^3+4x^2} \approx -0.000418987 \text{ at } x \approx -9.72596$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx 0.000517617 \text{ at } x \approx -9.09669$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx -0.000649883 \text{ at } x \approx -8.46727$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx 0.000831488 \text{ at } x \approx -7.83766$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx -0.00108785 \text{ at } x \approx -7.2078$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx 0.00146197 \text{ at } x \approx -6.57762$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx -0.00203047 \text{ at } x \approx -5.94699$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx 0.00293909 \text{ at } x \approx -5.31574$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx -0.00448874 \text{ at } x \approx -4.68357$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx 0.00737225 \text{ at } x \approx -4.04993$$

WA simply cannot find the general formula for the whole domain.

Always use your common sense and look at the graph as a check!

WA inflection points (sin(5x))/(3x^3+4x^2)

Inflection points between -100000 and 100000:

$$\frac{\sin(5x)}{3x^3+4x^2} = 0 \text{ at } x \approx -99999.4$$

$$\frac{\sin(5x)}{3x^3+4x^2} = 0 \text{ at } x \approx -10009.1$$

$$\frac{\sin(5x)}{3x^3+4x^2} = 0 \text{ at } x \approx -1000.91$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx 4.08858 \times 10^{-9} \text{ at } x \approx -99.9002$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx 0.0000628613 \text{ at } x \approx -9.39783$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx -0.000084315 \text{ at } x \approx -8.76745$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx 0.000115793 \text{ at } x \approx -8.13672$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx -0.00016351 \text{ at } x \approx -7.50554$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx 0.000238706 \text{ at } x \approx -6.87379$$

$$\frac{\sin(5x)}{3x^3+4x^2} \approx -0.000362922 \text{ at } x \approx -6.24125$$

Asymptotes

Horizontal asymptotes

(sin(5 x))/(3 x^3+4 x^2)->0 as x->±infinity

Vertical asymptotes

(sin(5 x))/(3 x^3+4 x^2)->±infinity as x->-4/3
(sin(5 x))/(3 x^3+4 x^2)->±infinity as x->0

Play with lots of examples.

Always try to check the answers against the Graph.

If you have someone to work with, give each other problems to try out.

Once, you study STEM subjects you will get many more examples that come up in all sorts of science and engineering math models.

Then, you will really appreciate the tool WA.

Tier 5 Calculus Lesson 3 Notes:

Derivative   Differential Calculus

Def. Let y = f(x) be a real valued function. f'(a) is defined to be the slope of the tangent line to the graph of f, at the point (a, f(a)). The tangent line is the unique straight line through this point "tangent" to the curve.

f'(a) is called the derivative of f at x = a. f'(x) is the derivative of f at x, and sometimes written dy/dx or y' or y'(x) in these latter cases it is understood that the derivative is evaluated at each x in the domain of f.

The slope of this tangent line can be computed as follows:

Definition of Infinitesimal, h. An infinitesimal, h, is a hyperreal number whose absolute value is less than any real number. Think of h as a very small number.

"std" means the unique real number part of the hyperreal number, e.g. std(3 + h) = 3

Modern (and original) definition of f'(a)

f'(a) = std[(f(a+h)-f(a))/h] where h is any nonzero infinitesimal.

Alternate definition of f'(a)

f'(a) = limit[(f(a+h)-f(a))/h] as h → 0 where h is any nonzero real number. This is the 19th century definition still used in many calculus books. The infinitesimal definition is the modern definition available since 1966, and also the original definition.

Also, we write $f'(a) = dy/dx|_{x=a} = y'(a) = D_x f(a)$

Rules for infinitesimals.

Let h and k be nonzero infinitesimals, a and b real numbers with b nonzero.

1. a + h, b + k are hyperreal numbers, and std(a+h)= a and std(b+k)=b
2. std(h) = 0
3. 0/h = 0
4. ah and bk and hk and h/b and h+k are all infinitesimal numbers
5. std[(a +h) o (b+k)] = std(a+h) o std(b+ k) = a o b where o is any operator + - X /
6. h^n is infinitesimal when n is integer > 1
7. $h^n < h^{n-1}$ for any integer n>1
8. a/h is infinite hyperreal, which we will not use.
9. h/k is indeterminate

Note: When computing derivative, f'(a) with a TI30Xa calculator just let a be any domain number, and h = .0000001. Then you will get the answer to a high degree of accuracy. Nothing is exact in the physical world. There are no irrational or infinitesimal numbers in the physical world, only rational approximations.

The Hyperreal Numbers can be constructed from the Real Numbers, or treated axiomatically. This was achieved by mathematicians in the 1960's. Hyperreal numbers are just as "real" as Real Numbers or Complex Numbers.

Examples

1. $y = f(x) = x^3$ Find f'(4), f'(a), and f'(x).

With TI30Xa Calculator

$f'(4) = std\{[(4+.000001)^3 - 4^3]/.000001\} =$
$std(48.00001) = 48$

$f'(a) = std\{[(a+h)^3 - a^3]/h\}$

$= std\{[a^3+3a^2h+3ah^2+h^3 - a^3]/h\}$

$= std\{3a^2 + 3ah + h^2\} = 3a^2$

$y = f'(x) = 3x^2$ by same calculation where x = a.

Note: $3(4)^2 = 48$.

Which was easier to compute?

2. $y = f(x) = x^n$, for any integer n >0, then
   $f'(x) = nx^{n-1}$

Demo: $f'(x) = std\{[(x+h)^n - x^n]/h\} =$

$$std\{nx^{n-1}+h(numbers)\}= nx^{n-1}$$

Note use of Binomial Theorem. The lead term x^n cancels with $-x^n$, and the next term is $nx^{n-1}h$ which then cancels with the h in the denominator and all of the remaining terms have higher powers of h.

3. y = f(x) = C for some constant number C.

Demo: $f'(x) = std\{[C - C]/h\} = 0$

4. y = f(x) = SIN(x)   Find f'(x) Use Radian Measure for x

Note: SIN(.000001)/.000001 = 1 = COS(.000001)

[(cos(.000001) – 1)/.000001] = 0

Assume the following which can be rigorously proven:

std{SIN(h)/h}= std{COS(h)}= 1

std{(COS(h) -1)/h} = 0

Demo: f'(x) = std{[SIN(x+h) – SIN(x)]/h} =

std{[SIN(x)COS(h) + SIN(h)COS(x) – SIN(x)]/h} =

std{[SIN(x)(COS(h)-1)]/h + [SIN(h)COS(x)/h]} =

SIN(x)std[(COS(h)-1)]/h] + COS(x)std[SIN(h)/h]

SIN(x)X0 + COS(x)X1 = COS(x)

Thus, if f(x) = SIN(x), then f'(x) = COS(x)

Find f'(2.4)

Std{[SIN(2.4 +.000001) – SIN(2.4)]/.000001}

= -.73739406,   COS(2.4) = -.737393716

Which is easier?

But, if you don't know a formula for the derivative you can always use the calculator for any function to find its derivative at a specific point in the domain.

5. If $y = f(x) = COS(x)$, then, $dy/dx = f'(x) = -SIN(x)$
Demo or Proof: Exercise

Find $f'(.87) =$
$std[(COS(.87+.000001) - COS(.87)/.000001] =$
$-.764329 = -SIN(.87)$

Which is easier?

The definition or knowing $f'(x) = -SIN(x)$?

6. If $y = f(x) = 1/x = x^{-1}$, then $f'(x) = -1/x^2 = -x^{-2}$

$f'(x) = std\{[1/(x+h) - 1/x]/h\} =$

$std\{[x-(x+h)]/(x+h)xh\} =$

$std\{-1/(x+h)x\} = -1/x^2 = -x^{-2}$

7. If $y = f(x) = 1/x^n = x^{-n}$ for n a positive integer,

Then $dy/dx = f'(x) = -n/x^{-n-1} = -nx^{-n-1}$

Demo or Proof:

Exercise, best done using the Chain Rule, which is in T5 C7 lesson, and #2 and #6 above.

So come back and study this after T5 C7.

$f(x) = (1/x)^n = x^{-n}$

So, $f'(x) = n(1/x)^{n-1}(-x^{-2}) = -n(x)^{1-n-2} = -nx^{-n-1}$

Calculus is like all math. How you solve a problem, or if you can solve it, depends on what tools are available to you.

In the next lesson you will learn to use the powerful 21st century tool, Wolfram Alpha to find derivatives.

T5 C3 Derivative Differential Calculus – Exercises

Q1. If y = f(x) is a real valued function, define f'(a).

Q2. What is the tangent line in calculus?

Q3. What is an infinitesimal, h?

Q4. What is the modern definition of f'(a)?

Q5. When computing derivative f'(a), you will get the answer to a high degree of accuracy with a TI30Xa calculator by using what value for your infinitesimal?.

Q6. What does "std" mean?

Q7. If $y = f(x) = 2x^3 + 5x^2 - 3x + 4$, find f'(x) and f'(3).

Q8. If $y = f(x) = 2/x^3 + 5/x^2 - 3/x + 4$, find f'(x) and f'(3).

Q9. If $y = f(x) = 2/x^3$, find f'(x) and f'(3).

Q10. If y = f(x) = tan(x), find f′(x) and f′(3π).

A1. f'(a) is defined to be the slope of the tangent line to the graph of f, at the point (a, f(a)).

A2. The tangent line is the unique straight line through this point "tangent" (perpendicular) to the curve.

A3. An infinitesimal, h, is a hyperreal number whose absolute value is less than any real number.

A4. f'(a) = std[(f(a+h)-f(a))/h] where h is any nonzero infinitesimal.

A5. 0.0000001

A6. "std" means the unique real number part of the hyperreal number.

A7. $f'(x) = 6x^2 + 10x - 3$, $f'(3) = 81$

A8. $f'(x) = -6/x^4 - 10/x^3 + 3/x^2$ or $f'(x) = (3x^2 - 10x - 6)/x^4$, $f'(3) = -0.111111$

A9. $f'(x) = -6/x^4$, $f'(3) = -0.0740740$

A10. $f'(x) = \sec^2(x)$, $f'(3\pi) = 1$

Tier 5 Calculus Lesson 4 Notes: Derivatives of Functions

This is a list of the derivatives of many common well known functions.

You can "derive" these derivatives using the definitions of the functions, various things we know about them, and facts about infinitesimals. It also is often best to utilize various "rules" governing derivatives we will list in the lessons T5 C6 and C7.

However, the quickest and easiest way to find the derivative of a function is to ask Wolfram Alpha, WA.

Simply type in derivative "formula for the function"

Example: derivative SIN(x) or derivative sin(x)

Answer COS(x) WA also gives the graph of the derivative and some other information.

Example: derivative x^n

Answer: nx^{n-1} Note: Use of Partial Derivative Notation.

You will probably want to learn the derivatives of many of the common functions.

Verify them all with WA.

If you are going to become a mathematician you will probably want to "prove" many of these formulas using infinitesimals, or the older techniques often taught in today's calculus textbooks.

If you are going to become a STE student, then you probably will just accept them as true thanks to WA.

List of Functions and their Derivative Function

$f(x)$	$f'(x)$ x is a real number in domain of f
x^n	nx^{n-1} n is any non zero real number
$SIN(x)$	$COS(x)$
$COS(x)$	$-SIN(x)$
$TAN(x)$	$SEC^2(x)$
e^x	e^x
$LN(x)$	$1/x = x^{-1}$
a^x	$LN(a)a^x$ $a > 0$
$Log_a(x)$	$1/xLog(a)$

Verify these with Wolfram Alpha

Commit these to memory if you are going to be tested on your ability to find derivatives.

Then, using the Rules of Differentiation you will be able to find the derivatives of much more complicated functions pretty easily.

For example $f(x) = [SIN(x^3) + e^{5x}]^{1/2}$

$f'(x) = .5[SIN(x^3) + e^{5x}]^{-1/2} [3x^2COS(x^3) + 5e^{5x}]$

Note: There was a small mistake in the notes in the video which has been corrected here. Do you see it?

This actually is very easy using the Rules which you will learn in the Lessons T5 C6 and C7.

T5 C4 Derivatives of Functions – Exercises

Use Wolfram Alpha to find the derivatives of the following functions:

Q1. $f(x) = 2(5x^2 + 3x - 7)^3$

Q2. $f(x) = (4x^3 - 2x^2 - x^2 + 8)/(3x^2 + 5x + 2)$

Q3. $f(x) = 3\sin(5e^{4x})$

Q4. $f(x) = \sin(\cos(x^2))$

Q5. $f(x) = \dfrac{(x-5)(2x+3)}{x^2+2x}$

Q6. $f(x) = \dfrac{\cos(2x^3+6x^2)}{4x^2}$

Q7. $f(x) = \dfrac{\tan(e^{6x\wedge 2})}{3x^3+4x^2}$

Q8. $f(x) = \sin(2x)\cos(3x^2)$

Q9. $f(x) = \dfrac{\sin(2x)}{\cos(3x^2)}$

Q10. $f(x) = \dfrac{(4x^3-3x^2+6x-3)}{\tan(3x)}$

A1. $\frac{d}{dx}\left(2\left(5x^2+3x-7\right)^3\right)=6\,(10x+3)\left(5x^2+3x-7\right)^2$

A2. $\frac{d}{dx}\left(\frac{4x^3-2x^2-x^2+8}{3x^2+5x+2}\right)=\frac{12x^4+40x^3+9x^2-60x-40}{\left(3x^2+5x+2\right)^2}$

A3. $\frac{d}{dx}\left(3\sin\left(5\,e^{4x}\right)\right)=60\,e^{4x}\cos\left(5\,e^{4x}\right)$

A4. $\frac{d}{dx}\left(\sin\left(\cos\left(x^2\right)\right)\right)=-2x\sin\left(x^2\right)\cos\left(\cos\left(x^2\right)\right)$

A5. $\frac{d}{dx}\left(\frac{(x-5)\,(2x+3)}{x^2+2x}\right)=\frac{11x^2+30x+30}{x^2\,(x+2)^2}$

A6. $\frac{d}{dx}\left(\frac{\cos\left(2x^3+6x^2\right)}{4x^2}\right)=-\frac{\left(6x^2+12x\right)\sin\left(2x^3+6x^2\right)}{4x^2}-\frac{\cos\left(2x^3+6x^2\right)}{2x^3}$

A7. $\frac{d}{dx}\left(\frac{\tan\left(e^{6x^2}\right)}{3x^3+4x^2}\right)=\frac{12\,e^{6x^2}\,x\sec^2\left(e^{6x^2}\right)}{3x^3+4x^2}-\frac{\left(9x^2+8x\right)\tan\left(e^{6x^2}\right)}{\left(3x^3+4x^2\right)^2}$

A8. $\frac{d}{dx}\left(\sin(2x)\cos\left(3x^2\right)\right)=2\cos(2x)\cos\left(3x^2\right)-6x\sin(2x)\sin\left(3x^2\right)$

A9. $\frac{d}{dx}\left(\frac{\sin(2x)}{\cos\left(3x^2\right)}\right)=2\cos(2x)\sec\left(3x^2\right)+6x\sin(2x)\tan\left(3x^2\right)\sec\left(3x^2\right)$

A10. $\frac{d}{dx}\left(\frac{4x^3-3x^2+6x-3}{\tan(3x)}\right)=\left(12x^2-6x+6\right)\cot(3x)-3\left(4x^3-3x^2+6x-3\right)\csc^2(3x)$

Tier 5 Calculus Lesson 5 Notes:

Application of Derivatives for Graphs

Graphing a Function, f(x)

1. One wants to plot the function over an appropriate range of domain values.

This has been very difficult classically. Graphing calculators made it much easier, and now, Wolfram Alpha makes it even easier. This you have already learned and calculus doesn't help much here.

2. One wants to determine the Roots, where $f(x) = 0$ and crosses the x axis. Again WA does this for you, and calculus won't help much.

3. One wants to determine all of the Asymptotes, where $f(x)$ approaches a vertical line as $x \to a$ or another function $g(x)$ as $x \to \infty$

These usually are determined algebraically, or with a tool like Wolfram Alpha. Calculus doesn't help much here either.

4. Increasing, Decreasing, and Concavity are obvious once one has graphed the function. However, there are tests for these at any specific point x = a using derivatives. See the Function Graph Term Sheet for Calculus

Classically, this was used to help construct the graphs manually.

5. Determine maxima, minima, and points of inflection.

Historically, Calculus was used to do this.

But, once again Wolfram Alpha does this for you as you have already learned.

Stationary(critical) points are where $f'(x) = 0$ and these are possible local maxima, minima or points of inflection.

6. Inflection points are where the concavity switches signs. Then $f''(x) = 0$, but not conversely.

WA does this for you automatically, as you have learned.

The conclusion we can draw is that using derivatives to graph functions and determine max, min, inflection points, and concavity are no longer the easy or best way to do it.

But, let's look at an example of what you could be taught in a traditional calculus course and

1. **How it would be done with WA, and then**
2. **How it would have been done historically with calculus prior to WA.**

Indeed, you can construct the graph and analyze any function with these five WA steps.

Many examples that arise in STEM are actually too difficult to even do in a reasonable length of time the classical way.

But, WA can handle virtually any function.

This is possible because your problem is being dealt with by a large bank of computers running a very sophisticated program, Mathematica.

Example: $f(x) = x^5 - 3x^4 - 5x^3 + 15x^2 + 4x - 12$

A polynomial we will first graph and analyze with WA

1 Plot x^5 -3x^4 -5x^3 +15x^2 +4x -12

2 Roots x^5 -3x^4 -5x^3 +15x^2 +4x -12

Maxima x^5 -3x^4 -5x^3 +15x^2 +4x -12

Minima x^5 -3x^4 -5x^3 +15x^2 +4x -12

3 Stationary Points

x^5 -3x^4 -5x^3 +15x^2 +4x -12

4 Inflection Points

x^5 -3x^4 -5x^3 +15x^2 +4x -12

5 Asymptotes

x^5 -3x^4 -5x^3 +15x^2 +4x -12

How would you do this with calculus and derivatives the classical way?

There is no easy way to find the roots with calculus. You would have to use algebra algorithms, if you could, or use approximation techniques like Newton's method.

Next you try to find the values of x where the first derivative $f'(x) = 0$. These are called critical points and might be maxima, or minima, or points of inflection. Additional tests are required to determine these.

This may give you an idea of where the roots are located approximately so you can use Newton's Method.

Then, you find the points where the second derivative, $f''(x) = 0$. These might be points of inflection. But additional tests are required to find out.

Once you know the maxima, minima you can graph the function. The roots help make this graph more accurate as do the points of inflection.

At any particular point, a, you may determine if the function is increasing or decreasing by $f'(a)>0$ for increasing, and $f'(a)<0$ for deceasing.

Similar tests are available for concavity tests.

$f''(a)>0$ concave up, $f''(a)<0$ concave down.

$f''(a)=0$ might be a point of inflection. It could be a max or min like $f(x) = x^4 + 5$ where $f''(x) = 12x^2 = 0$ for $x = 0$ and is a minimum.

Now, let's do the above example using the classical techniques with derivatives.

$f(x) = x^5 -3x^4 -5x^3 +15x^2 +4x -12$

Find roots first. Not easy unless problem rigged up to use rational roots. Must use algebraic algorithm or approximation technique like Newton's Method.

Roots x^5 -3x^4 -5x^3 +15x^2 +4x -12

Now let's use some calculus.

1. **Find the derivative, $f'(x)$. This is easy**

It is $f'(x) = 5x^4 - 12x^3 - 15x^2 +30x +4$

2. **Now find the roots of $f'(x)$ for critical points. Often this is not easy. Must use some algebraic algorithm or Newton's Method.**

Note: the roots of the derivative are the possible stationary points.

Roots of Derivative

x^5 -3x^4 -5x^3 +15x^2 +4x -12

3. **Apply additional tests to determine maxima and minima. Second derivative test, or direct test by evaluating f(x) on both sides of the critical point.**

4. **Take the second derivative f''(x). Easy**
   $f''(x) = 20x^3 – 36x^2 – 30x + 30 = 0$

5. **Find the inflection points by finding the roots of the second derivative. Finding the roots of f''(x) may not be easy.**

Roots of Second Derivative

x^5 -3x^4 -5x^3 +15x^2 +4x -12

6. **Then apply additional tests to determine if which roots are points of inflection.**

7. **Find the Asymptotes. Calculus does not apply here.**

Actually, it is even harder for many examples from STEM subjects to use calculus.

T5 C5 Function Graph Term Sheet for Calculus

Function f : $y = f(x)$, $x \varepsilon D \le R$, $y \varepsilon R$

D = Domain = Set of numbers f(x) is defined for.

R = Real Numbers, ≤ "is a subset of", ε "is contained in"

Graph of f, Set of (x, f(x)) in plane, x ε D

<u>Terms</u> describing f, at (a, f(a)) for any number a ε D

Defined	**f(a) is defined, a ε D**
Continuous	**f(a) = Limit of f(x) as $x \to a$**
Smooth	**$f'(a)$ exists, $f'(a)$ = Limit $[f(a+h)-f(a)]/h$ as $h \to 0$**
Increasing	**$f'(a) > 0$**
Decreasing	**$f'(a) < 0$**
Flat	**$f'(a) = 0$**
Max (Local)	**$f(a-h)<f(a)>f(a-h)$, Small $h>0$**
Min (Local)	**$f(a-h)>f(a)<f(a-h)$, Small $h>0$**
Concave Up	**$f''(a) > 0$ or f' is increasing at a**
Concave Down	**$f''(a) < 0$ or f' is decreasing at a**

Inflection Point $f''(a) = 0$ or f' is going from + to – or visa versa usually, but not always.

Vertical Asymptote (Usually a not in Domain)

$f(x) \to \infty$ or $f(x) \to -\infty$ as $x \to a$ from left or right

Asymptote g(x) when $x \to \infty$ or $x \to -\infty$

$f(x) \to g(x)$ as $x \to \infty$ or $x \to -\infty$

Difficult to graph and analyze a function with traditional calculus techniques and, in fact, impossible for many actual functions arising in STEM. Always Easy with Wolfram Alpha.

T5 C5 Application of Derivatives for Graphs Exercises

Q1. What is the value of f’(x) at a stationary (critical) point?

Q2. Stationary points can indicate on a graph the location of what?

Q3. What happens an inflection point (f’’(a)=0)?

Q4. When f’’(a) > 0 the graph is concave ______ at point a.

Q5. When f’’(a) < 0 the graph is concave ______ at point a.

Q6. What is an asymptote?

Q7. What is a root?

Q8. What is a domain?

Q9. What does the symbol “$\leq$” mean?

Q10. What does the symbol “$\in$” mean?

A1. 0

A2. Local maxima, local minima, or inflection points

A3. Concavity switches signs

A4. Up

A5. Down

A6. An asymptote is where $f(x) \rightarrow$ a vertical line as x approaches a or another function $g(x)$ as $x \rightarrow \infty$.

A7. A root is where $f(x) = 0$ and crosses the x axis.

A8. A domain is a set of numbers $f(x)$ is defined for.

A9. Is a subset of

A10. Is contained in

Tier 5 Calculus Lesson 6 Notes: Derivative Rules

Let f(x) and g(x) be real valued functions where

$f'(x)$, $g'(x)$, and $f'(g(x))$ all exist, then

I. $[cf(x)]' = cf'(x)$ **for any constant c.**

II. $[(f(x) + g(x))]' = f'(x) + g'(x)$

III. $[f(x)g(x)]' = f(x)g'(x) + f'(x)g(x)$ **Leibniz Rule**

IV. $[(f(g(x))]' = f'(g(x))g'(x)$ **Chain Rule**

V. $[f(x)/g(x)]' = [f'(x)g(x) - f(x)g'(x)]/[g(x)]^2$
Quotient Rule

VI. $[1/f(x)]' = -f'(x)/[f(x)]^2$, $f(x) \neq 0$

VII. $[f^{-1}]'(x) = 1/f'(f^{-1}(x))$ **Inverse Function Rule**

This is when $y = f^{-1}(x)$ IFF $x = f(y)$

With these Rules one can find the derivative of most complicated functions which are built up from the common well known functions by simply applying the Rules sequentially in a "nested" way.

No VII. Will be fully discussed in a T5 C11.

These Rules can be demonstrated most easily with infinitesimal arguments, which can be made fully rigorous.

This is how our ancestors originally discovered them and understood them.

Here are a few demonstrations and you may do some for yourself, especially if you are contemplating a math major or becoming a mathematician.

However, if you are a STE student, you might just want to remember the rules for test taking purposes and practice for the tests.

Today, you will use a tool like WA to actually find derivatives and their roots and graphs.

Demo of I

$[cf(x)]' = std[(cf(x+h)-cf(x))/h]$

$= std[c(f(x+h) - f(x))/h] = cf'(x)$

Examples: Verify with WA

$[6\sin(x)]' = 6\cos(x)$ $[7x^4]' = 28x^3$ $[8.3e^x]' = 8.3e^x$

Demo of II.

$[f(x) + g(x)]' = std\{([f(x+h) + g(x+h)) - (f(x)+g(x))]/h\}$

$= std\{[f(x+h) - f(x)]/h + [g(x+h)-g(x)]/h\}$

$= std\{[f(x+h) - f(x)]/h\} + std\{[g(x+h)-g(x)]/h\}$

$= f'(x) + g'(x)$

Examples: Verify with WA

$[\sin(x) + 5e^x + 2x^3]' = \cos(x) + 5e^x + 6x^2$

$[\ln(x) - \tan(x)]' = 1/x - \sec^2(x)$

Demo of III. The Liebniz Rule is a little counter intuitive. You might think $[f(x)g(x)]' = f'(x)g'(x)$. But a quick example or two will show this is not true. Here is the type of argument Leibniz, Newton, and Euler would have used. Only, they would have used Δx instead of h, and assumed that Δx behaved the way we say infinitesimals like h behave. The modern arguments from non-standard analysis look a lot like Euler's arguments. You try it first!

$[f(x)g(x)]' = \text{std}\{[f(x+h)g(x+h) - f(x)g(x)]/h\} =$

$\text{std}\{[f(x+h)g(x+h)-f(x+h)g(x)+f(x+h)g(x)- f(x)g(x)]/h\}$

$= \text{std}\{[f(x+h)[g(x+h)-g(x)]]/h\} +$

$\quad \text{std}\{[[f(x+h)- f(x)]g(x)]/h\}$

$= \text{std}\{f(x+h)\}X\text{std}\{[g(x+h)-g(x)]/h\} +$

$\quad \text{std}\{[f(x+h)- f(x)]/h\}X\text{std}\{g(x)\}$

$= f(x)g'(x) + f'(x)g(x)$

Examples: Verify with WA

$[\sin(x)\cos(x)] = \sin(x)(-\sin(x)) + \cos(x)\cos(x) =$
$\cos^2(x) - \sin^2(x) = \cos(2x)$

$[x^3e^x]' = x^3e^x + 3x^2e^x = e^x x^2(x + 3)$

$[\sin(x)\ln(x)]' = \sin(x)/x + \cos(x)\ln(x)$

The Chain Rule will be discussed in some more depth in T5 C7.

However, we will now apply it to Demo VI.

$$[1/f(x)]' = -f'(x)/[f(x)]^2 \ , f(x) \neq 0$$

Demo. **Define** $h(x) = 1/f(x)$

Suppose $y = f(x)$, **and** $g(y) = 1/y = y^{-1}$

Then $h(x) = g(f(x))$ **and**

$h'(x) = g'(f(x))f'(x)$ **by the Chain Rule.**

But, $g'(y) = -1y^{-2} = -1/y^2$ **and thus**

$g'(f(x)) = -1/[f(x)]^2$

And it follows: $h'(x) = -f'(x)/[f(x)]^2 = [1/f(x)]'$

Exercise: You should now be able to derive V. using this result and the Leibniz Rule.

Examples for V. Verify with WA

$[\sin(x)/\cos(x)]' = [\cos(x)\cos(x) - \sin(x)(-\sin(x)]/\cos^2(x)$

$= [\cos^2(x) + \sin^2(x)]/\cos^2(x) = 1/\cos^2(x) = \sec^2(x)$

So, if $f(x) = \tan(x)$, **then** $f'(x) = \sec^2(x)$

$[x^3/\sin(x)]' = [3x^2\sin(x) - x^3\cos(x)]/\sin^2(x)$

$= x^2[3\csc(x) - x\cot(x)\csc(x)] = x^2[3 - x\cot(x)]\csc(x)$

Observation:

Do you see how Algebra is so necessary in deriving these Rules.

I don't see any easy way to derive the Leibniz Rule or the Chain Rule from just Geometry. If you do, please email me and let me know.

Yet, ultimately these Derivative Rules are what we need to fully understand the geometry of functions.

These are the tools our ancestors used for about 300 years which created modern science and technology.

Ironic how Algebra and Geometry compliment and support each other, isn't it?

Calculus is just an extension of Algebra and Geometry which lets us understand the behavior many functions by understanding their rates of change.

But, the best is yet to come.

It turns out that derivatives are the key to understanding how functions can accumulate things, like the area under a graph.

This will culminate in what is called the Fundamental Theorem of Calculus, which is arguably the foundation of all of our modern technology and science models.

T5 C6 – Derivative Rules Homework

For the following equations, which rule would you use to find the derivative, f'(x)? Find the derivative for each function manually and verify with Wolfram Alpha.

Q1. $f(x) = 3x^4 – 6x^3 + 8x^2 + 5x - 9$

Q2. $f(x) = \ln(x)\tan(x)$

Q3. $f(x) = 3\sin(x)$

Q4. $f(x) = (3x^3 + 5x^2 – x + 5)/(4x^4 – 2x^3)$

Q5. $f(x) = 3/\cos(x)$

Q6. $f(x) = 3\sin(2e^{2x})$

Q7. $f(x) = 4\cos(x)$

Q8. $f(x) = \ln(x)5e^{3x}$

Q9. $f(x) = \sin(x)/3x^2$

Q10. $f(x) = 3\cos(5x^2)$

A1. Rule II; $\frac{d}{dx}\left(3x^4 - 6x^3 + 8x^2 + 5x - 9\right) = 12x^3 - 18x^2 + 16x + 5$

A2. Rule III, or Leibniz Rule; $\frac{d}{dx}(\log(x)\tan(x)) = \frac{\tan(x)}{x} + \log(x)\sec^2(x)$

A3. Rule 1; $\frac{d}{dx}(3\sin(x)) = 3\cos(x)$

A4. Rule V, or Quotient Rule;

$$\frac{d}{dx}\left(\frac{3x^3 + 5x^2 - x + 5}{4x^4 - 2x^3}\right) = \frac{-6x^4 - 20x^3 + 11x^2 - 42x + 15}{2(1 - 2x)^2 x^4}$$

A5. Rule VI; $\frac{d}{dx}\left(\frac{3}{\cos(x)}\right) = 3\tan(x)\sec(x)$

A6. Rule IV, or Chain Rule; $\frac{d}{dx}\left(3\sin\left(2\boldsymbol{e}^{2x}\right)\right) = 12\boldsymbol{e}^{2x}\cos\left(2\boldsymbol{e}^{2x}\right)$

A7. Rule I; $\frac{d}{dx}(4\cos(x)) = -4\sin(x)$

A8. Rule III, Leibniz Rule; $\frac{d}{dx}\left(\log(x)\times 5\boldsymbol{e}^{3x}\right) = \frac{5\boldsymbol{e}^{3x}(3x\log(x) + 1)}{x}$

A9. Rule V, or Quotient Rule; $\frac{d}{dx}\left(\frac{\sin(x)}{3x^2}\right) = \frac{x\cos(x) - 2\sin(x)}{3x^3}$

A10. Rule IV, or Chain Rule; $\frac{d}{dx}\left(3\cos\left(5x^2\right)\right) = -30x\sin\left(5x^2\right)$

Tier 5 Calculus Lesson 7 Notes: Chain Rule

Demo of IV.

Assume Let $f(x)$ and $g(x)$ be functions and assume $f'(x)$ and $g'(x)$ and $f'(g(x))$ all exist.

Then, $[f(g(x)]' = f'(g(x))g'(x)$

Note: $g(x+h)-g(x) = k \neq 0$, or $g(x+h) = g(x) +k$

where h, k are non-zero infinitesimals since $g'(x)$ exists and g is continuous at x

Note: $std[g(a+h)] = g(a)$ if g is continuous at x = a.

Demo:

$$[(f(g(x))]' = std\{[f(g(x+h))-f(g(x))]/h\} =$$

$$std\{[f(g(x+h))-f(g(x))]/[g(x+h)-g(x)]x$$

$$[g(x+h)-g(x)]/h\}$$

$$= std\{[f(g(x)+k) - f(g(x))]/k\}\ std\{[g(x+h)-g(x)]/h\}$$

$$=f'(g(x))g'(x)$$

One may apply the Chain Rule in a nested fashion.

If $H(x) = f(g(k(x))$, then

$$H'(x) = f'(g(k(x))g'(k(x))k'(x)$$

$$= k'(x)g'(k(x))f'(g(k(x))$$

Example:

$H(x) = [Cos(x^2 +3x)]^3$

$H'(x) = 3[\quad]^2[-(2x +3)Sin(\quad)]$

$= 3[Cos(x^2 +3x)]^2 X [-(2x +3)Sin(x^2 +3x)]$

WA Derivative (cos(x^2 +3x)^3 Watch the ( )

WA Derivative cos(x^2 +3x)^3

$H(x) = cos[(x^2 + 3x)^3]$

$H'(x) = -sin[(x^2 + 3x)^3]3(x^2 + 3x)^2(2x + 3)$

WA Derivative cos((x^2 + 3x)^3)

Exercise

$H(x) = [ln(sin(5x))]^3$

$H'(x) = 3[ln(sin(5x))]^2 X[1/sin(5x)]X[cos(5x)]X5$

$= 15[ln(sin(5x))]^2 Xcot(5x)$

Derivative (ln(sin(5x)))^3

Exercise

$f(x) = [SIN(x^3) + e^{5x}]^{1/2}$

$f'(x) = .5[SIN(x^3) + e^x]^{-1/2} [3x^2COS(x^3) + 5e^{5x}]$

Exercise

$F(x) = e^{-x2}$ $\qquad$ $F'(x) = -2x\, e^{-x2}$

WA Derivative e^-(x^2)

Observation:

Do you see how Algebra is so necessary in deriving these Rules.

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T5 C7 Chain Rule Exercises

For the following functions use the Chain Rule to find the derivative, and then use Wolfram Alpha to find the derivative of each function. Remember, pay careful attention to where you place ( ), and make sure Wolfram Alpha is interpreting the function in the way you intended.

Q1. $f(x) = \cos(3x^2 - 2x)^3$

Q2. $f(x) = \cos((2x - 3x^2)^3)$

Q3. $f(x) = \cos(\ln(3x^2))$

Q4. $f(x) = \cos(\ln(3x)^2)$

Q5. $f(x) = \cos(\ln(3x))^2$

Q6. $f(x) = \sin(3e^{5x\wedge 2})$

Q7. $f(x) = \cos(4e^{\sin(2x)})$

Q8. $f(x) = (\sin(e^{x\wedge 2}))^3$

Q9. $f(x) = \sin(4e^{\cos(2x)})$

Q10. $f(x) = \tan(\ln(2x^2 - 6x))$

A1. $$\frac{d}{dx}\left(\cos^3\left(3x^2 - 2x\right)\right) = -3\,(2 - 6x)\sin\left(2x - 3x^2\right)\cos^2\left(2x - 3x^2\right)$$

A2. $$\frac{d}{dx}\left(\cos\left(\left(2x - 3x^2\right)^3\right)\right) = 6\,(2 - 3x)^2\,x^2\,(3x - 1)\sin\left((2 - 3x)^3\,x^3\right)$$

A3. $$\frac{d}{dx}\left(\cos\left(\log\left(3x^2\right)\right)\right) = -\frac{2\sin\left(\log\left(3x^2\right)\right)}{x}$$

A4. $$\frac{d}{dx}\left(\cos\left(\log^2(3x)\right)\right) = -\frac{2\log(3x)\sin\left(\log^2(3x)\right)}{x}$$

A5. $$\frac{d}{dx}\left(\cos^2(\log(3x))\right) = -\frac{\sin(2\log(3x))}{x}$$

A6. $$\frac{d}{dx}\left(\sin\left(3\,e^{5x^2}\right)\right) = 30\,e^{5x^2}\,x\cos\left(3\,e^{5x^2}\right)$$

A7. $$\frac{d}{dx}\left(\cos\left(4\,e^{\sin(2x)}\right)\right) = -8\,e^{\sin(2x)}\sin\left(4\,e^{\sin(2x)}\right)\cos(2x)$$

A8. $$\frac{d}{dx}\left(\sin^3\left(e^{x^2}\right)\right) = 6\,e^{x^2}\,x\sin^2\left(e^{x^2}\right)\cos\left(e^{x^2}\right)$$

A9. $$\frac{d}{dx}\left(\sin\left(4\,e^{\cos(2x)}\right)\right) = -8\sin(2x)\,e^{\cos(2x)}\cos\left(4\,e^{\cos(2x)}\right)$$

A10. $$\frac{d}{dx}\left(\tan\left(\log\left(2x^2 - 6x\right)\right)\right) = \frac{(4x - 6)\sec^2\left(\log\left(2x^2 - 6x\right)\right)}{2x^2 - 6x}$$

Tier 5 Calculus Lesson 8 Notes: Implicit Differentiation

Suppose you have a relationship between two variables, x and y, expressed by an equation $F(x,y)=0$.

Example $F(x, y) = x^3 - 6xy + y^3 = 0$ (Folium of Descartes)

There will be a set of points in the x,y plane, that satisfies this equation, called its graph. In general, this will not be the graph of a function, but rather the graphs of several functions "glued together".

Constructing this graph can be quite challenging without a graphing calculator or computer algebra system or Wolfram Alpha, WA.

WA1 Plot x^3 - 6xy + y^3 = 0 from x = -4 to 4

A point (a, b) on this graph may have a tangent line.

Example, (2.7, 1.376) is such a point, i.e. $F(2.7, 1.376) = 0$

Finding such a point in the first place is also sometimes very challenging. Of course, WA makes it easy.

WA2 Solve for y when x = 2.7, x^3 - 6xy + y^3 = 0

There will be a tangent line to this graph passing through this point, (a, b).

The problem is to find the equation of this tangent line.

Of course, WA makes it easy.

WA3 Tangent line at x = 2.7, x^3 - 6xy + y^3 = 0

OK. WA makes this very easy.

We can even see that the slope of this tangent line is 1.294 and its y intercept is -2.118

We also saw two other points where x = 2.7, and their tangent lines too. No extra charge.

But, in the old days, and some current calculus books, there is no WA tool, so this is how they had to do it.

The first approach, which is often impossible, is to find a function y = f(x) whose graph is the same as F(x,y)=0 at (a,b) = (a,f(a)).

Then f'(a) will be the slope of the tangent line and

y = f'(a)(x – a) + f(a) is the equation.

Except for simple problems in calculus books like conic sections this can be quite difficult or impossible.

Just look at this example:

WA4 Solve for x, x^3 – 6ax +a^3

WA treats x and y as variables and a as a constant

So substitute y for x and x for a and you see the function.

Clearly, not easy to derive.

So there needed to be an easier way for our ancestors to find f'(a) without finding f(x) explicitly.

Implicit Differentiation is what they used.

Simply assume y is a function of x, and apply the rules of derivatives to F(x, y) = 0 and differentiate and solve for y'.

Example:

$F(x, y) = x^3 – 6xy + y^3 = 0$ (Folium of Descartes)

$3x^2 – 6xy' – 6y + 3y^2y' = 0$ (Liebniz and Chain Rules)

Solve for $y' = (6y - 3x^2) / (3y^2 – 6x) = (x^2 – 2y) / (2x – y^2)$

We want f'(2.7) at the point (2.7, 1.376)

f'(2.7) = y'(2.7) = (-13.614) / (-10.520) = 1.294

So, equation of the tangent line is:

y – 1.376 = 1.294(x – 2.7) or y = 1.294x – 2.118

Yaa! Check against WA3 answer.

This was easy to do manually with a calculator.

Of course WA would do if for us to.

WA5 Derivative x^3 – 6xy + y^3 = 0

And, then to actually calculate y'(2.7)

WA6 (x^2 – 2y) / (2x – y^2) at x = 2.7 and y = 1.375

But, with WA this would be the long way to do it.

Example: Derivative, y', of y = 1 / x or xy – 1 = 0

We learned earlier the answer is $y' = -1/x^2$

Let's do it another way with implicit derivatives.

F(x,y) = xy -1 = 0 Assume y is function of x

Thus, xy' + y = 0 using the Liebniz Rule

Thus, $y' = -y/x = -1/x^2$

WA7 Derivative xy -1 = 0

$y' = -y/x = -1/x^2$ since y = 1 / x

Example: Find the tangent line to the ellipse whose equation is $3x^2-3x+5xy+8y^2 = 15$ at the point (2.3, .408)

WA8 Plot 3x^2 -3x +5xy + 8y^2 – 15 = 0

WA9

Tangent line when x = 2.3, 3x^2 -3x +5xy + 8y^2 – 15 = 0

Now you can do this with implicit differentiation the old way.

Implicit differentiation is also very useful for finding the derivatives of inverse functions.

Example: $y = \log(x)$ or $x = e^y$

$1 = e^y y'$ or $y'(x) = 1 / e^y = 1 / x$

WA10 Derivative x - e^y = 0

$y' = e^{-y} = 1 / e^y = 1 / x$

Example Derivative of $y = \sin^{-1}(x)$ or $x = \sin(y)$

$1 = \cos(y)y'$

$y'(x) = 1 / \cos(y) = 1 / [(1 - \sin^2(y)]^{1/2} = 1 / (1 - x^2)^{1/2}$

WA11 Derivative x – sin(y) = 0

But, of course, WA would just give it to us directly.

WA12 derivative inverse sin(x)

So, we see Implicit Differentiation was a very potent tool for our ancestors, and you may see some STEM topic treated with it. So, you know what it is.

However, you will probably solve any real problem you run into with WA since it is so much more powerful and easy to use.

Tier 5 Calculus Lesson 8 Exercises: Implicit Differentiation

Q1. What is implicit differentiation?

Q2. Why is implicit differentiation used?

Q3. What is a tangent line, and what does it represent?

Q4. If a function is represented by $f(x)$, how is the slope represented at a point “a”?

Q5. Graph the function $2x^2 + 3xy + 3y^3 = 0$, and solve for y when $x=-0.2$

Q6. Graph the function $2x^2 + 3x + 3xy + 3y^2 - 4y = 17$, and solve for y when $x=-5$.

Q7. Graph the function $5x^2 - 2x - 4xy + 3y^2 + y = 8$, and solve for y when $x=1$.

Q8. Find the tangent line when $x = -0.2$ for the function $2x^2 + 3xy + 3y^3 = 0$.

Q9. Find the tangent line when $x = -5$ for the function $2x^2 + 3x + 3xy + 3y^2 - 4y = 17$

Q10. Find the tangent line when $x = 1$ for the function $5x^2 - 2x - 4xy + 3y^2 + y = 8$.

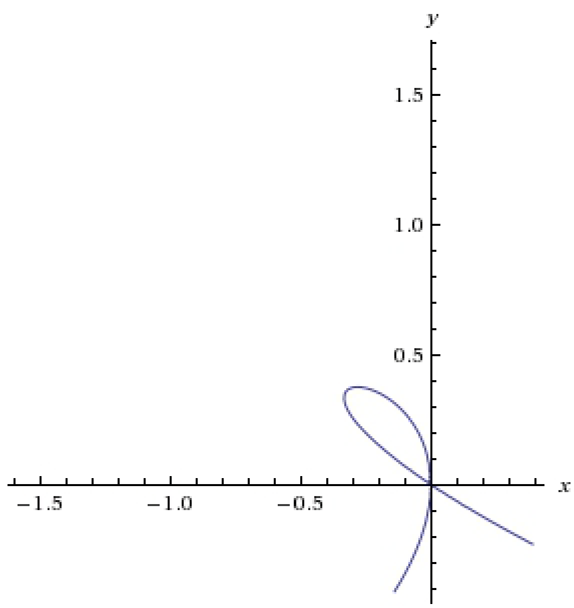
A1. Assume y is a function of x, and apply the rules of derivatives to F(x, y) = 0 and differentiate with respect to x, and then solve for y'. The answer will be a formula involving both x and y.

A2. It may be difficult or impossible to express y as a function of x, and thus an explicit derivative cannot be found.

A3. A tangent line is a straight line that touches a function at only one point. The tangent line represents the instantaneous rate of change of the function at that one point.

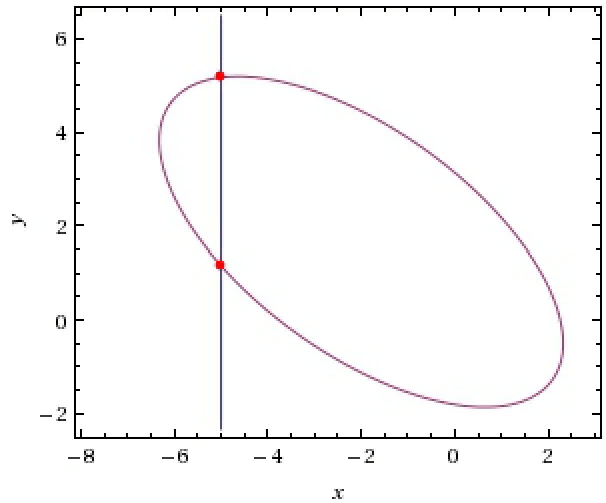
A4. f'(a)

A5.



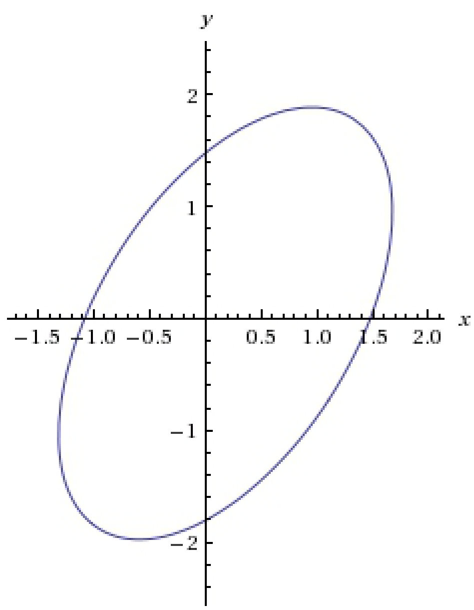
y = -0.503006, 0.150315, 0.352691

A6.



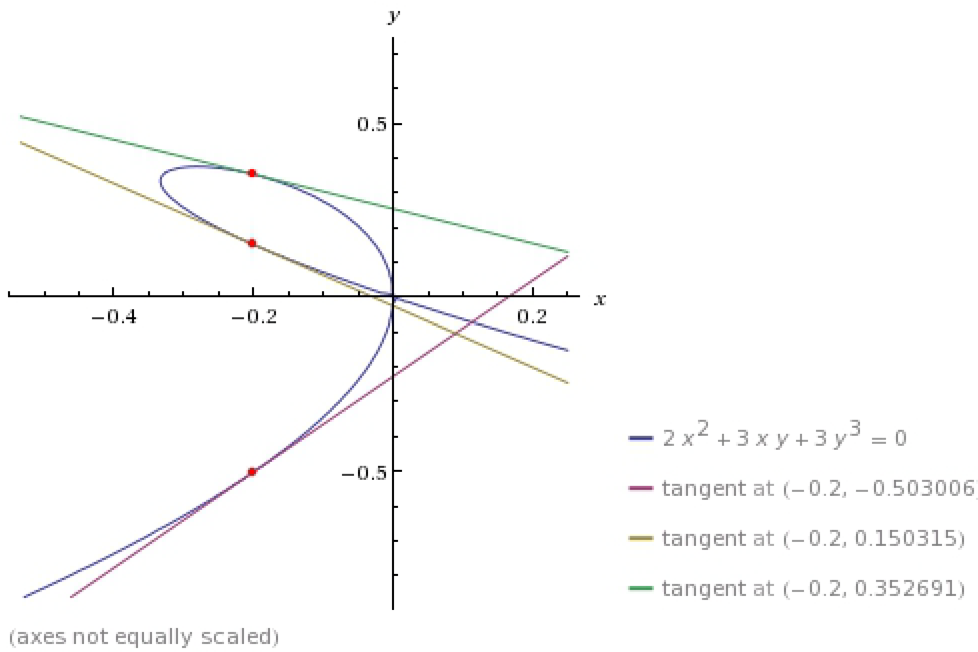
y = 1.15973, 5.17360

A7.



y = -0.884437, 1.88444

A8.

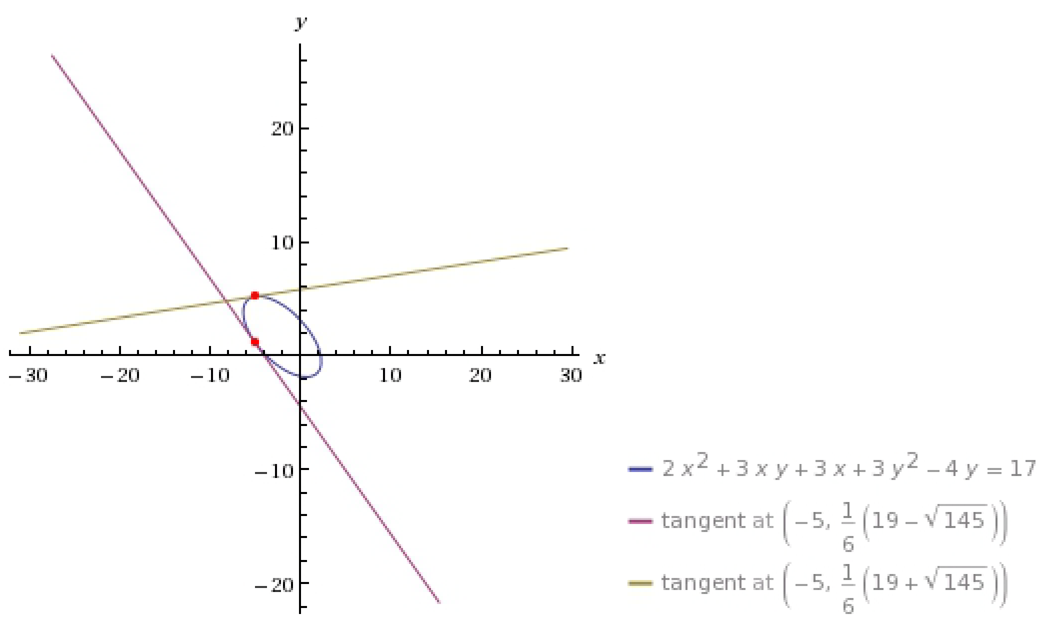


tangent at (x, y) = (-0.2, -0.503006) : y = 1.37677 x-0.227653

tangent at (x, y) = (-0.2, 0.150315) : y = -0.880011 x-0.0256874

tangent at (x, y) = (-0.2, 0.352691) : y = 0.25334-0.496754 x

A9.



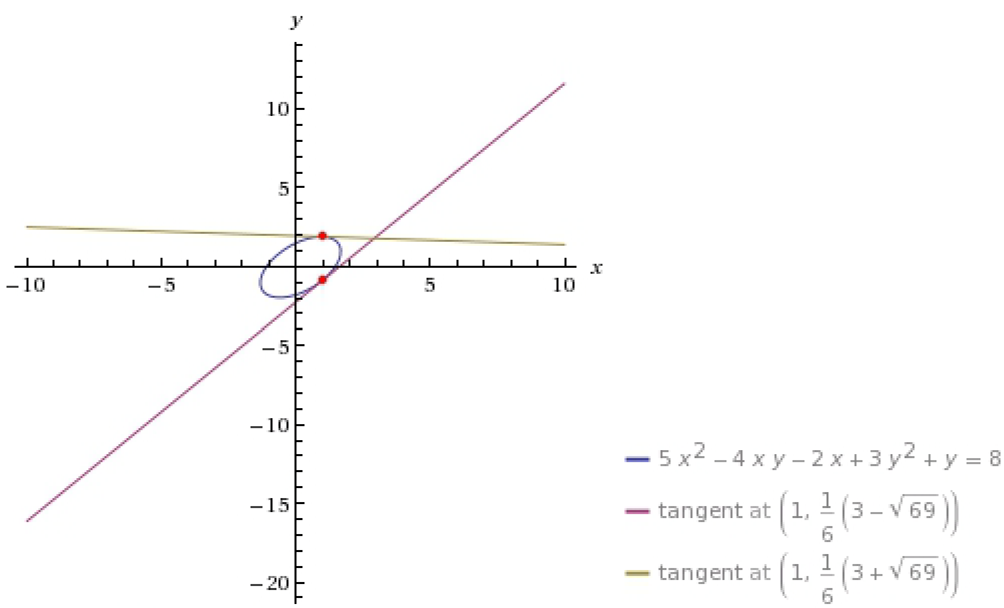
tangent at $(x, y) = \left(-5, \frac{1}{6}\left(19 - \sqrt{145}\right)\right)$:

$$y = \frac{1}{87}\left(58 - 37\sqrt{145}\right) - \frac{\left(17 + \frac{1}{2}\left(\sqrt{145} - 19\right)\right)x}{\sqrt{145}}$$

tangent at $(x, y) = \left(-5, \frac{1}{6}\left(19 + \sqrt{145}\right)\right)$:

$$y = \frac{\left(17 + \frac{1}{2}\left(-19 - \sqrt{145}\right)\right)x}{\sqrt{145}} + \frac{1}{87}\left(58 + 37\sqrt{145}\right)$$

A10.



(axes not equally scaled)

tangent at $(x, y) = \left(1, \frac{1}{6}\left(3 - \sqrt{69}\right)\right)$:

$$y = \frac{2\left(4 + \frac{1}{3}\left(\sqrt{69} - 3\right)\right)x}{\sqrt{69}} + \frac{1}{138}\left(-23 - 35\sqrt{69}\right)$$

tangent at $(x, y) = \left(1, \frac{1}{6}\left(3 + \sqrt{69}\right)\right)$:

$$y = \frac{1}{138}\left(35\sqrt{69} - 23\right) - \frac{2\left(4 + \frac{1}{3}\left(-3 - \sqrt{69}\right)\right)x}{\sqrt{69}}$$

Tier 5 Calculus Lesson 9 Notes: Related Rates

If f(t) is a function of time, t, we say f′(t) is its "rate of change with respect to time". And, at t = a we say f′(a) is its rate of change at time t = a. If f(t) is measured in distance unit ft and time t is in seconds, we say f′(t) is measured in ft/sec. Or, it could be m/hr if f(t) is measured in miles, m, and time, t, in hours.

Suppose we have two functions f(t) and g(t) and their derivatives f′(t) and g′(t), and suppose that we have a linear equation with these four functions. For example $f'(t)f(t) = - g(t)g'(t)$.

Then if we know three of them, we can easily find the fourth one. Why?

Remember, if you have four unknowns U,X,Y,Z related by a linear equation, and if you know any three of them, then you can solve for the fourth one easily.

For example. Suppose you know g(t) = 6ft, g′(t) = 2ft/sec, and f(t) = 8ft then

$f'(t) = - g(t)g'(t)/f(t) = -6ft x 2ft'sec/8ft = -3/2ft/sec$

Notice, In this example you don't know what t is actually equal to.

So, the challenge is always to find a linear equation which relates the four functions, f(t), f′(t), g(t), and g′(t).

This can be accomplished in various ways depending on the specific problem being addressed. You will have to use the geometry of the situation to relate f(t) and g(t) usually.

Example 1: A ladder is leaning against a wall. The ladder is 10 ft long. The base is being moved away from the wall at the rate of 2ft/sec. How fast will the top of the ladder be moving down the wall when the base is 6 ft from the wall?

Draw a picture and set up the functions and the equation that relates them. x(t) is the distance of the base from the wall, and y(t) is the height of the ladder up the wall, both at time t. At any time t, by the P.T. we know an equation that relates them

$x(t)^2 + y(t)^2 = 10^2$

and $x'(t) = 2$ **ft/sec and** $x(t) = 6$ **ft.**

Now, $y(t) = 8$ **ft by the P.T. and we want to know** $y'(t)$**.**

Note: This is not a linear equation, and $x'(t)$ **and** $y'(t)$ **are not in this equation. So, what to do?**

Apply implicit differentiation to this equation w.r.t. t

$x(t)^2 + y(t)^2 = 10^2$

$2x(t)x'(t) + 2y(t)y'(t) = 0$ **and solve for** $y'(t)$

Thus, $y'(t)= -x(t)x'(t)/y(t)$ **ft/sec [ftX(ft/sec)/ft = ft/sec]**

$y'(t) = -6\text{X}2/8$ **ft/sec = -3/2 ft/sec = -1.5 ft/sec**

Note we did not have to calculate t.

Question? The base is moving out at a constant rate of 2ft/sec. What about the top moving down?

Is it moving down at a constant rate? $y'(t)$ **?**

Note: $y(t) = [1 - x^2(t)]^{1/2}$ by the P.T. So, as $x(t)$ gets larger $y(t)$ gets smaller.

Does it go faster or slower as you pull the ladder away from the wall i.e. is $y'(t)$ increasing or decreasing?

Examine $y'(t)= -x(t)x'(t)/y(t)$ ft/sec

As $x(t)$ increases, $y(t)$ decreases, so $y'(t)$ increases

Example 2. In the above situation how fast will the top of the ladder be moving down the wall when the base is 7 ft from the wall?

$x(t) = 7$, and thus, $y(t) = (100 - 49)1/2 = 7.14$

$y'(t) = -7X2/7.14$ ft/sec = 1.96 ft/sec > 1.5 ft/sec

So, it's going faster.

9 ft from the wall the top is going down -9x2/4.36 ft/sec = 4.1 ft/sec , so faster yet. Note: $4.36 = (100 - 81)^{1/2}$

Example 3. If the ladder is coming down the wall at the rate of -4 ft/sec when $y(t) = 3$ ft, how fast is the base moving away from the wall?

Now, we need to find $x'(t)$.

We know $x(t) = (100 - 9)^{1/2}$ =9.54 ft by the geometry.

when and $y(t) = 3$ ft and we know $y'(t) = -4$ft/sec

Solve $y'(t)= -x(t)x'(t)/y(t)$ for $x'(t)$

$x'(t) = -y'(t)y(t)/x(t)$ [(ft/sec)ft/ft) = ft/sec]

$x'(t) = -(-4\text{ft/sec})X3\text{ft}/9.54\text{ft} = 1.3$ ft/sec

Always check the units to catch a mistake in the equation.

There are many ways one might relate the two functions. It all depends on the situation. You need to come up with a relationship between f(t) and g(t), and then differentiate to create a linear relationship between all four functions, f(t), g(t), f′(t), and g′(t). Then you need to find the values of three of them and calculate the value of the fourth unknown one.

Sometimes f(t) and g(t) are related via a third function h where f(t) = h(g(t)).

Now you use the Chain Rule:

f′(t) = h′(g(t))g′(t).

So you need to know g(t), h′(g(t)), and g′(t) to find f′(t), or f′(t), h′(g(t)) and g′(t) to find g(t), etc. [see end of next lesson, C10, for further explanation.]

Example 4: A sphere has radius r and volume V, and the sphere is inflating at the rate of 100 in^3/sec and we want to know how fast the radius is increasing when the radius equals 25 in.

So, let r(t) and V(t) be the volume and radius at time, t.

We are given V′(t) = 100 in^3/sec and r(t) = 25 in.

What is r′(t)?

We know V(t) = 4/3∏r(t)^3 by the volume formula.

So, V′(t) = 4∏r(t)^{2}r′(t) or r′(t) = V′(t)/ 4∏r(t)^2

So, r′(t) = (100 in^3/sec)/4∏25^{2}in^2 = 1/25∏ in/sec

Note: We did not have to calculate the time t.

Example 5. Suppose the radius of a sphere is increasing at the rate of 2 in/sec when it is equal to 12 in. How fast will the volume of the sphere be increasing?

We know $r'(t) = 2$ in/sec and $r(t) = 12$.

We know $V(t) = 4/3\pi r(t)^3$ by the volume formula.

So, $V'(t) = 4\pi r(t)^2 r'(t)$

So, $V'(t) = 4\pi X 12^2(in^2)X2in/sec = 1152\pi\ in^3/sec$

Another type of rate problem.

Suppose you know $r'(t)$ and $r(0)$ and $V(0)$, and you wanted to know $V(5)$.

Then you would need to figure out $r(5)$ since

$V(5) = 4/3\pi r(5)^3$

This is a solvable problem, but not with what you have learned so far in this course.

It will be solvable when you learn the extension of Calculus called Differential Equations.

Tier 5 Calculus Lesson 9 Exercises: Related Rates

Q1. What are related rates?

Q2. If you are given two functions, f(t) and g(t), and their derivatives, f'(t) and g'(t), and the following linear equation represents the relationship between them: f(t)f'(t) = g(t)g'(t), find f(t) if f'(t)=15 in/sec, g(t)=180 in, and g'(t)=40 in/sec.

Q3. If you are given two functions, f(t) and g(t), and their derivatives, f'(t) and g'(t), and the following linear equation represents the relationship between them: f(t)/f'(t) = g(t)/g'(t), find f(t) if f'(t)=65 miles/hr, g(t)=300 miles, and g'(t)=75 miles/hr.

Q4. You are traveling at 70 miles/hr and realize that the rest of your trip will take you 3.75hr, but you need to reach your destination in 3.5hr. What does your rate of speed (mi/hr) need to be to reach your destination on time?

Q5. A cube has a length s (lengths on all sides are equal) and volume V, and the cube is increasing in volume at the rate of 96 in^3/sec. We want to know how fast the length is increasing when the length equals 4 in.

Q6. A sphere has radius r and volume V, and the radius of the sphere is increasing at the rate of 5 in/sec. We want to know how fast the volume of the sphere is increasing when the radius equals 15 in.

Q7. A sphere has radius r and surface area A, and the radius of the sphere is increasing at the rate of 5 in/sec. We want to know how fast the surface area of the sphere is increasing when the radius equals 15 in.

Q8. Two people are standing 20 ft. apart. One of them walks north at a rate of 3 ft/sec. At what rate is the distance between them changing when the person walking is 8 ft. from her starting point?

Q9. A tank of water in the shape of an upside-down cone is leaking water at a rate of 3ft^3/hour. At what rate is the radius of the surface of the water changing when the radius r = 8ft and the height h of the water is 15 ft?

Q10. A cylindrical tank has a radius of 4 ft. If water is being added to the tank with a garden hose at a rate of 0.75 ft^3/min, what is the rate of change of the height of the water?

A1. Related rates involve finding the rate at which one quantity changes when the rate of change for another quantity is known and the two rates are in some way related.

A2. 480 in

A3. 260 miles

A4. 75 miles/hr

A5. $V(t) = s^3$

$V'(t) = 3s^2s'(t)$

$96 = 3(4)^2s'(t)$

$s'(t) = 2$ in/sec

A6. $V(t) = 4/3\pi r(t)^3$

$V'(t) = 4\pi r^2r'(t)$

$V'(t) = 4\pi(15)^2(5)$

$V'(t) = 4500\pi$, or 14137.17, in^3/sec

A7. $A(t) = 4\pi r^2$

$A'(t) = 8\pi rr'(t)$

$A'(t) = 8\pi(15)(5)$

$A'(t) = 600\pi$, or 1884.96, in^2/sec

A8. $x^2 + y(t)^2 = h(t)^2$

$20^2 + 8^2 = h(t)^2$

$h(t)^2 = 464$ ft^2

$h(t) = 21.54$ft

$20^2 + y(t)^2 = h(t)^2$

$0 + 2y(t)y'(t) = 2h(t)h'(t)$

$y(t)y'(t) = h(t)h'(t)$

$(8)(3) = (21.54)\ h'(t)$

$h'(t) = 1.114$ ft/sec

A9. $V(t) = 1/3\pi r(t)^2 h(t)$ and we are given $r(t) = 8$ft and $V'(t) = 3\text{ft}^3/\text{hr}$ and $h(t) = 15$ft and we want to know $r'(t)$

$V'(t) = 1/3\pi[2r(t)r'(t)h(t) + r(t)^2 h'(t)]$ Note: Use Leibniz Rule and Chain Rule

Oops. We have two unknowns, $r'(t)$ and $h'(t)$

So, let's start over and eliminate $h'(t)$ by using what we know about the geometry.

$V(t) = 1/3\pi r(t)^2 h(t)$

We know $h(t)/r(t) = 15/8$

Therefore $h(t) = (15/8)r(t)$, and thus

$V(t) = (5/8)\,\pi r^3(t)$ since

$V(t) = (1/3)\pi r(t)^2 h(t) = (1/3)\pi(15/8)r(t)^3 = (5/8)\pi r(t)^3$

$V'(t) = (5/8)\pi 3r(t)^2 r'(t)$ and we are given $V'(t) = 3\text{ft}^3/\text{hr}$ and $r(t) = 8$ft

$3 = (15/8)\,\pi 8^2 r'(t) = 120\pi r'(t)$

$r'(t) = 3/120\pi = .00796$ ft/hr

A10. $V(t) = \pi r^2 h$

$V'(t) = \pi r^2 h'(t)$

$0.75 = \pi(4^2)h'(t)$

$h'(t) = 0.046875/\pi$, or 0.01492, ft/min

T5 C10a Inverse Functions Basics

Lesson T5 C10 starts out with the following.

Let $f(x)$ be a function which is 1 to 1 on a domain D.

That is: if $f(a) = f(b)$, then $a = b$

One horizontal line will only intercept the graph of f at only one point.

Define a function f^{-1} whose domain is the Range of f as follows: $y = f^{-1}(x)$ if and only if $x = f(y)$

f^{-1} is called the inverse function of f

Note: the -1 is NOT an exponent, just a symbol

The graph of f^{-1} is the reflection of the graph of f in the 45^o line, graph of $y = x$.

This is because (a,b) is the reflection of (b,a) in this line.

In this lesson T5 C10a let us elaborate on this. Inverse functions are very important, and sometimes looking at them geometrically is very helpful.

First, let's consider the fact that (a,b) and (b,a) are reflections in the 45^o line which is the graph of the line

$y = f(x) = x$

Second, let's be sure we understand what it means for a function $y = f(x)$ to be 1 to 1 on a Domain D.

For example, $y = f(x) = x^2$ is not 1 to 1 over the domain of all the real numbers. However, it is 1 to 1 over the domain of the non-negative real numbers, and also over the domain of non-positive real numbers or of any subdomains of these two domains.

Example, $y = \sin(x)$ is 1 to 1 over the domain

$-Pi/2<x<Pi/2$ or many other similar subdomains.

Now let's look at some specific examples of inverse functions. The simplest are linear functions whose graphs are straight lines. Consider $y = f(x) = mx + b$, where m is the slope and b is the y-intercept. What is its inverse function?

By definition: $y = f^{-1}(x)$ means $x = f(y)$ or $x = my + b$

Solving for y, $y = (1/m)x - b/m = f^{-1}(x)$

Example, if $f(x) = 5x+3$, then $f^{-1}(x) = (1/5)x - 3/5$

WA1 inverse function 5x + 3

Example, if $f(x) = 2x-1$, then $f^{-1}(x) = (1/2)x + ½$

WA2 inverse function 2x – 1

What if $m = 1/m$, then $m = +1$ or -1

If $m = +1$, the line is parallel to the 45^o line and its inverse is just a parallel line on the other side.

If $m = -1$, the line is perpendicular to the 45^o line and is its own inverse

If $m = 0$, $1/m = \infty$ and we have a horizontal line and vertical line as inverses.

In general, if the graph of a function is symmetrical about the 45^o line, then it is its own inverse.

Example, $y = f(x) = 1/x$ is its own inverse.

WA3 inverse function 1/x

Some non linear examples:

$y = f(x) = x^2$ for $x \geq 0$ Clearly, $y = f^{-1}(x) = x^{1/2}$ Why?

$x = y^2$ and thus $y = x^{1/2}$

WA 4 inverse function x^2

What about $y = f(x) = e^x$?

In this case the inverse function is so important it has been given a name, ln(x) or log(x) to the base e.

So $y = \log(x)$ iff $x = e^y$ [iff means "if and only if"]

That is why we say log(x) is an exponent.

WA 5 inverse function e^x

What about y = f(x) = sin(x) ?

This is very important inverse function, but it only is give the name $\sin^{-1}(x)$ or Arcsin(x)

Of course we must restrict the domain as indicated above to $-Pi/2 < x < Pi/2$ or something similar. The domain of the arcsin(x) is $-1< x< 1$ which was, of course, the range of sin(x). The range of Arcsin(x) is $-Pi/2 < x < Pi/2$

WA 6 Plot sin(x), inverse sin(x)

WA 7 Plot sin(x), inverse sin(x) from x =-1 to 1

You can do similar things with any function to find its inverse, and plot it.

Now, let's address the question of what would be the derivative of $f^{-1}(x)$ from a geometric perspective. In C10 we will approach it using tools from calculus.

I will do this graphically here.

Remember that the inverse function of a straight line

$f(x) = mx + b$ is $f^{-1}(x) = (x - b) / m$

Also, remember the derivative of any function $g(x)$ is just the slope of the tangent line at $(x, g(x))$ or $g'(x)$

$(f^{-1})'(x) = 1 / f'(f^{-1}(x))$ for any x.

Let's see this geometrically here, and in C10 we will derive this using calculus.

T5 C10a Inverse Functions Basics Exercises

Q1. A function f(x) will see its inverse function as a reflection about which line?

Q2. If f(x) is a function which is 1 to 1 on Domain D, what does that mean?

Q3. What is the inverse function of $y = 6x + 5$? Graph both in Wolfram Alpha.

Q4. What is the inverse function of $y = 3x - 4$? Graph both in Wolfram Alpha.

Q5. What is the inverse function of $y = 2x^2 - 3$? Graph both in Wolfram Alpha.

Q6. In Q5, name two domains to make the function 1 to 1.

Q7. What is the inverse function of $y = x^3$? Graph both in Wolfram Alpha.

Q8. In Q7, does the graph of the function f(x) need to be divided into domains to make the function 1 to 1? If so, name a domain.

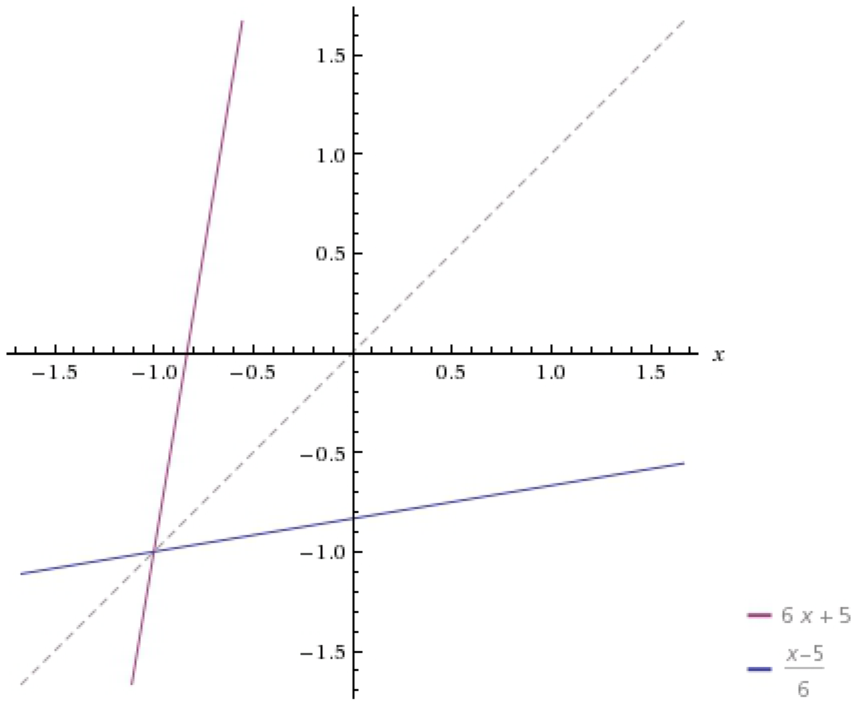
Q9. What is the inverse function of $y = \cos(x)$? Graph both in Wolfram Alpha.

Q10. In Q9, does the graph of the function f(x) need to be divided into domains to make the function 1 to 1? If so, name a domain.

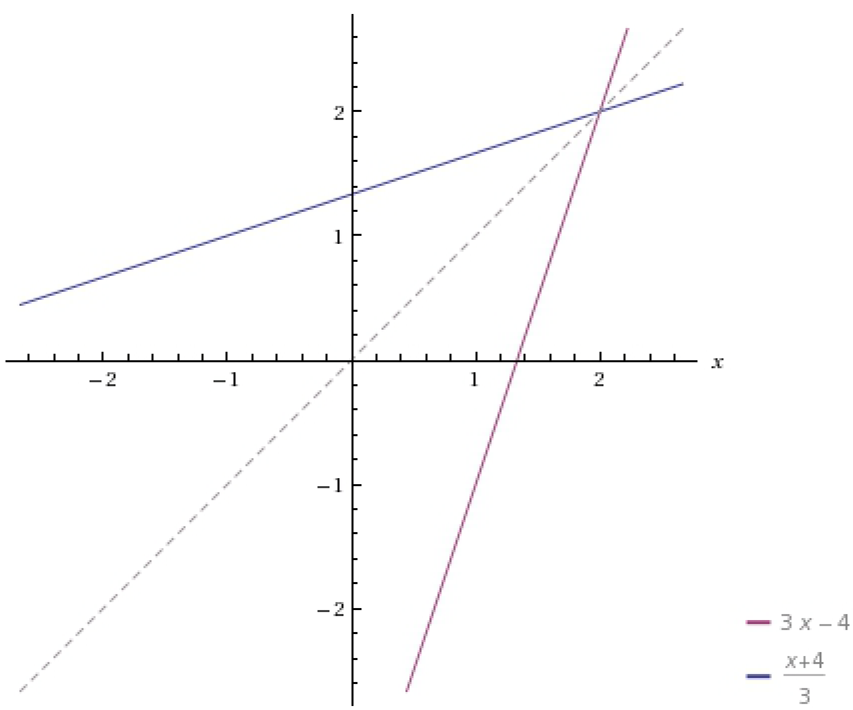
A1. The 45° line

A2. One horizontal line will only intercept the graph of f(x) at only one point. For each value of x in the Domain D, there will be only one answer f(x).

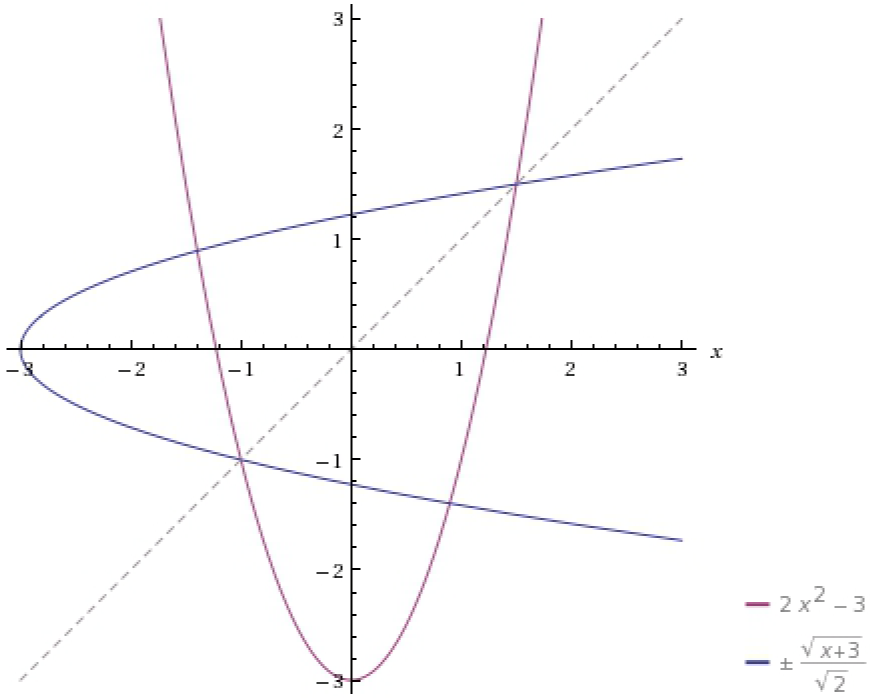
A3. $x = y/6 - 5/6$



A4. $x = y/3 + 4/3$

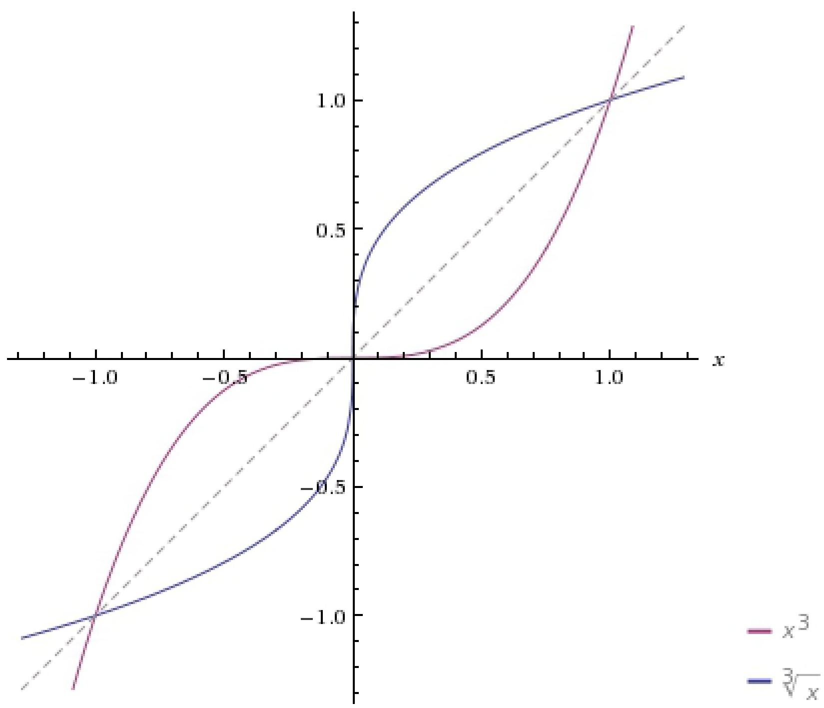


A5. $x = (y/2 + 3/2)^{1/2}$



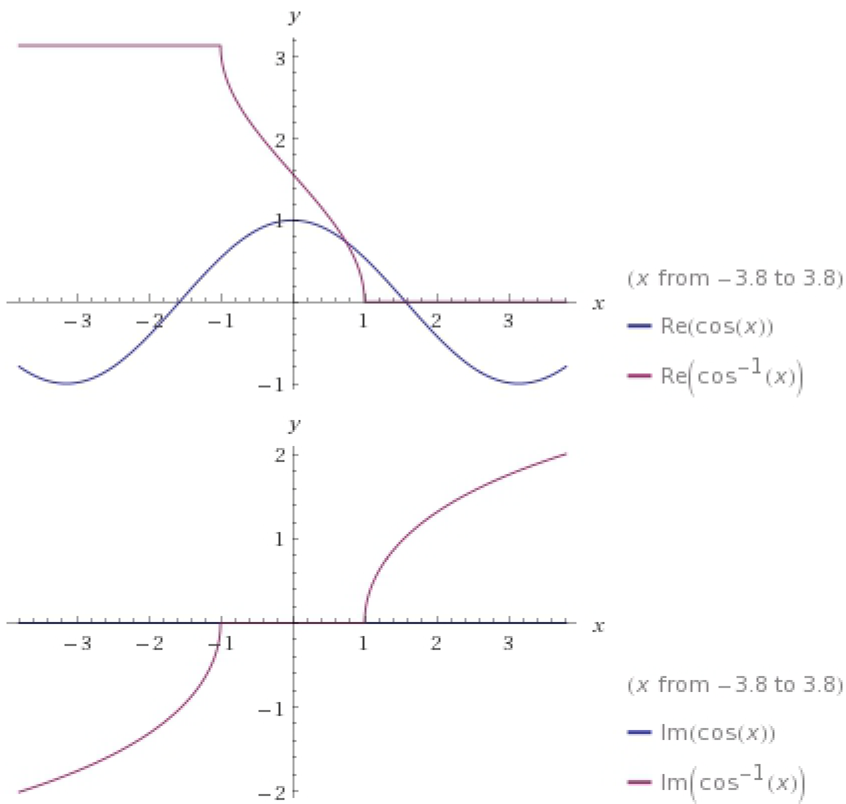
A6. Non-negative real numbers, non-positive real numbers

A7. $x = (y)^{1/3}$

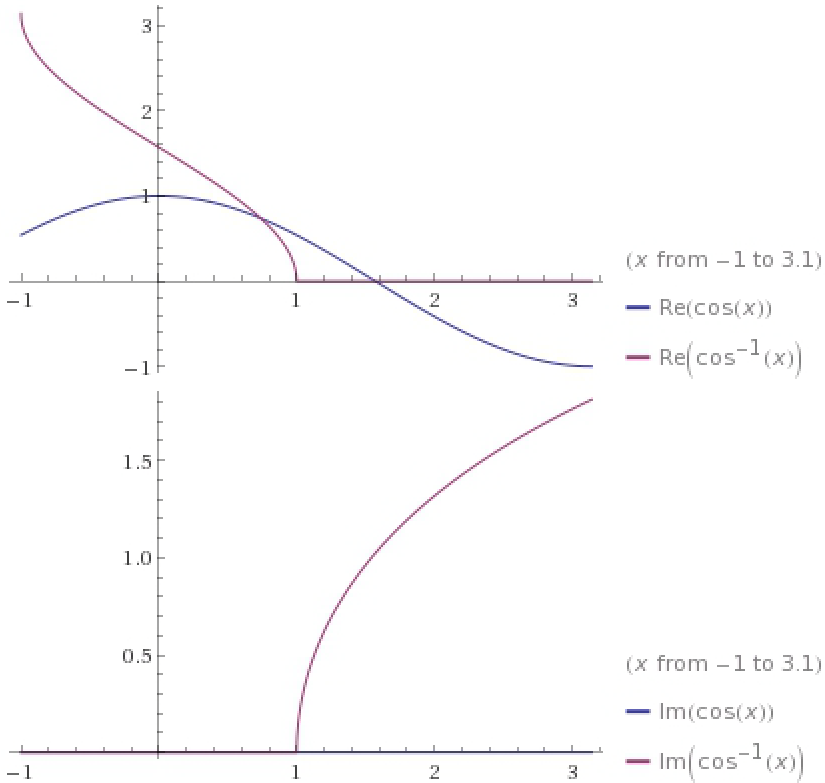


A8. No, at no point can a horizontal line be drawn through the graph of the function f(x) and cross through multiple points.

A9. $y = \cos^{-1}(x)$

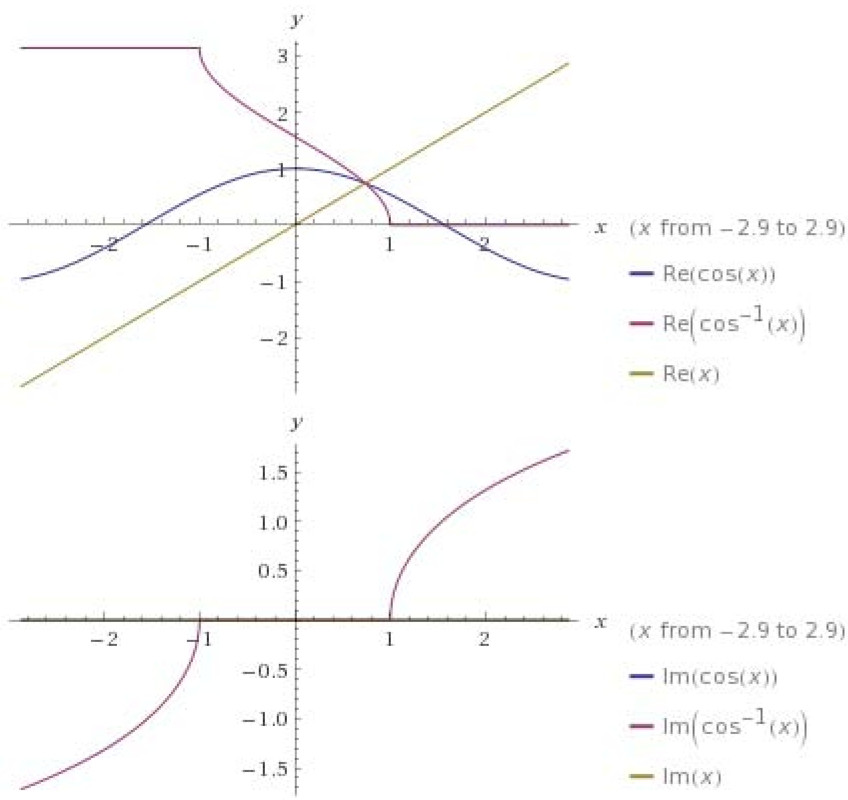


But this answer is not quite correct. The problem is that the domain of the function, Cos(x) is 0 to π and its range is -1 to 1, so the domain of the inverse Cos is -1 to 1. The best we can do is to include both domains in the smallest possible interval which is -1 to π. The corrected graph:



Plotting also y = x helps to see the mirror effect.

WA plot cos(x), inverse cos(x), x



A10. Yes; 0 to π

T5 C10 Inverse Functions

Let f(x) be a function which is 1 to 1 on a domain D.

That is: if f(a) = f(b) , then a = b

One horizontal line will only intercept the graph of f at one one point.

Define a function f^{-1} whose domain is the Range of f as follows: $y = f^{-1}(x)$ if and only if $x = f(y)$

f^{-1} is called the inverse function of f

Note: the -1 is NOT an exponent, just a symbol

The graph of f^{-1} is the reflection of the graph of f in the 45^{o} line, graph of y = x.

This is because (a,b) is the reflection of (b,a) in this line.

Now the question is: "What is the derivative of f^{-1} ?

$y = f^{-1}(x)$ is equivalent to $x = f(y)$

Implicitly Differentiate x = f(y) to get

$1 = f'(y)y' = f'(f^{-1}(x))(f^{-1})'(x)$

So, $$(f^{-1})'(x) = 1 / f'(f^{-1}(x))$$

Example 1: If $f(x) = x^2$, then $y = f^{-1}(x) = x^{1/2}$

Why? $y = f^{-1}(x)$ means $x = f(y) = y^2$ or $y = x^{1/2}$

So $y = f^{-1}(x) = x^{1/2}$

Now, $f'(x) = 2x$ so according to our formula:

$(f^{-1})'(x) = 1/f'(f^{-1}(x)) = 1/2\ x^{1/2} = (1/2)x^{-1/2}$

Of course, we already know $y' = (1/2)x^{-1/2}$ from long ago.

Now from first principles, what is the derivative of f^{-1} or $y' = (f^{-1})'(x)$?

Here again: $y = (f^{-1})(x)$ means $x = f(y) = y^2$

So, $1 = 2yy'$ and, thus, $y' = 1/2y$

$(f^{-1})'(x) = y' = 1/2y = 1/2x^{1/2} = (1/2)x^{-1/2}$

Example 2: $y = \cos^{-1}(x)$ is inverse of $\cos(x)$

and $[\cos(x)]' = -\sin(x)$

So $[\cos^{-1}(x)]' = y' = 1 / -\sin(\cos^{-1}(x))$

Now consider the definitions of sin and cos in trig

$\sin^2(a) + \cos^2(a) = 1$, so $\sin(a) = [1 - \cos^2(a)]^{1/2}$

So, let $a = \cos^{-1}(x))$ and we get

$\sin(\cos^{-1}(x)) = [1 - \cos^2(\cos^{-1}(x)))]^{1/2} = [1 - x^2]^{1/2}$

since by definition $\cos(\cos^{-1}(x)) = x$

So $y' = (\cos^{-1})'(x) = -1 / [1 - x^2]^{1/2}$

Verify with WA

WA1 derivative inverse cos(x)

Example 3. $y = \tan^{-1}(x)$ and $[\tan(x)]' = \sec^2(x)$

$[\tan^{-1}(x)]' = 1 / \sec^2(\tan^{-1}(x)] = 1 / (1 + x^2)$

Since $1 + \tan^2(a) = \sec^2(a)$

WA derivative inverse tan(x)

Example 4

$y = \sec^{-1}(x) \qquad x = \sec(y)$

$[\mathrm{Sec}(x)]' = [1/\cos(x)]' = -(-\sin(x))/\cos^2(x) = \tan(x)\sec(x)$

$y' = 1/\tan(\sec^{-1}(x))\sec(\sec^{-1}(x))$

$\tan^2(a) = \sec^2(a) - 1$ or $\tan(a) = [\sec^2(a) - 1]^{1/2}$

$y' = 1/(x^2 - 1)^{1/2}x = 1/(1 - 1/x^2)^{1/2}x^2$

WA derivative inverse sec(x)

Example 5

Related Rates Example revisited.

"Now you use the Chain Rule:

$f'(t) = h'(g(t))g'(t)$.

So you need to know g(t), h'(g(t)), and g'(t) to find f'(t), or f'(t), h'(g(t)) and g'(t) to find g(t), etc."

In this last case, how will you find g(t)?

Clearly you will have to use the inverse of h' to do so.

$g(t) = (h')^{-1}(g(t))$

T5 C10 Inverse Functions Exercises

Note: This will be a challenging set of problems. It took me several hours to fully understand them and "master" Wolfram Alpha. As you will see starting in Q3 , it is best to use WA in two steps. But, you also have two manual ways of doing the problems. First, you find $f^{-1}(x)$ and differentiate it directly. Second, you can use the formula from Q2.

Q1. If a function is defined as f(x), what is $f^{-1}(x)$?

Q2. What is the general form for the derivative of the inverse function $(f^{-1})'(x)$ if the function is f(x)?

Q3. Given the function $y = f(x) = x^3$, what is the inverse function $f^{-1}(x)$, and what is the derivative of the inverse function $(f^{-1})'$?

Q4. Given the function $y = f(x) = 4x^2$, what is the inverse function $f^{-1}(x)$, and what is the derivative of the inverse function $(f^{-1})'$?

Q5. Given the function $y = f(x) = 8x^3$, what is the inverse function $f^{-1}(x)$, and what is the derivative of the inverse function $(f^{-1})'$?

Q6. Given the function $y = f(x) = \sin(x)$, use Wolfram Alpha to find the derivative of the inverse function $(f^{-1})'$. Note: $\sin^{-1}(x) = \arcsin(x)$ by definition

Q7. Given the function $y = f(x) = \sin^2(x)$, use Wolfram Alpha to find the derivative of the inverse function $(f^{-1})'$.

Q8. Given the function $y = f(x) = \sin(x^2)$, use Wolfram Alpha to find the derivative of the inverse function $(f^{-1})'$.

Q9. Given the function $y = f(x) = \sin(\cos(x))$, use Wolfram Alpha to find the derivative of the inverse function $(f^{-1})'$.

Q10. Given the function $y = f(x) = \cos(\sin(x))$, use Wolfram Alpha to find the derivative of the inverse function $(f^{-1})'$.

Q11. Given the function $y = f(x) = e^{\sin(x)}$, use Wolfram Alpha to find the derivative of the inverse function $(f^{-1})'$.

Q12. Given the function $y = f(x) = \cos(\log(x))$, use Wolfram Alpha to find the derivative of the inverse function $(f^{-1})'$.

A1. The inverse function of f(x)

A2. $(f^{-1})'(x) = 1/\ f'(f^{-1}(x))$

A3. $y = f(x) = x^3$, $f'(x) = y' = 3x^2$

$y = f^{-1}(x) \Longleftrightarrow x = f(y) \Longleftrightarrow x = y^3 \Longleftrightarrow y = x^{1/3} = f^{-1}(x)$

Direct $y' = 1/3x^{-2/3}$

Formula $(f^{-1})'(x) = 1/f'(f^{-1}(x)) = 1/3(\ x^{1/3}\ )^2 = 1/3x^{2/3} = (1/3)x^{-2/3}$

WA inverse x^3 Answer: $x^{1/3}$

WA derivative x^(1/3)

NOTE: Usually it is best to use WA in two steps. First find the inverse function and second evaluate the derivative of the inverse function.

A4. $(f^{-1})'(x) = (1/4)x^{-1/2}$

A5. $(f^{-1})'(x) = (1/6)x^{-1/3}$

A6. $(f^{-1})'(x) = 1/(1 - x^2)^{1/2}$

A7. WA inverse function sin^2(x)

$(f^{-1})(x) = \pm\sin^{-1}(x^{1/2})$

WA derivative sin^(-1)(x^(1/2))

$$\frac{d}{dx}\left(\sin^{-1}(\sqrt{x})\right) = \frac{1}{2\sqrt{-(x-1)\,x}}$$

A8. WA inverse function sin(x^2)

$(f^{-1})(x) = \pm(\sin^{-1}(x))^{1/2}$

WA derivative (sin^(-1)(x))^(1/2)

$$\frac{d}{dx}\left(\sqrt{\sin^{-1}(x)}\right) = \frac{1}{2\sqrt{1-x^2}\ \sqrt{\sin^{-1}(x)}}$$

A9. WA inverse function sin(cos(x))

$(f^{-1})(x) = \pm(\cos^{-1}(\sin^{-1}(x))$

WA derivative (cos^(-1)(sin^(-1)(x))

$$\frac{d}{dx}\left(\cos^{-1}\left(\sin^{-1}(x)\right)\right) = -\frac{1}{\sqrt{1-x^2}\,\sqrt{1-\sin^{-1}(x)^2}}$$

A10. WA inverse function cos(sin(x))

$(f^{-1})(x) = \pm(\sin^{-1}(\cos^{-1}(x))$

WA derivative sin^(-1)(cos^(-1)(x))

$$\frac{d}{dx}\left(\sin^{-1}\left(\cos^{-1}(x)\right)\right) = -\frac{1}{\sqrt{1-x^2}\,\sqrt{1-\cos^{-1}(x)^2}}$$

A11. WA inverse function e^(sin(x))

$(f^{-1})(x) = \sin^{-1}(\log(x))$

WA derivative sin^(-1)(log(x))

$$\frac{d}{dx}\left(\sin^{-1}(\log(x))\right) = \frac{1}{x\sqrt{1-\log^2(x)}}$$

A12. WA inverse function cos(log(x))

$(f^{-1})(x) = \pm e^{\arccos(x)}$ Reminder: arccos = $\cos^{-1}$

WA derivative e^(cos^(-1)(x))

$$\frac{d}{dx}\left(e^{\cos^{-1}(x)}\right) = -\frac{e^{\cos^{-1}(x)}}{\sqrt{1-x^2}}$$

Tier 5 Calculus Lesson 11 Notes: Series Expansions

Polynomials are particularly easy to deal with using calculus since they are just a sum of powers of x.

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_2x^2 + a_1x + a_0$$

$$P'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + 2a_2x + a_1$$

So, if you can approximate a function f(x) with a polynomial P(x), then you can easily differentiate it, and more importantly as we will learn, integrate it.

We can also write a polynomial in reverse order

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n$$

One would think that the higher the order, n, of the polynomial the better approximation of f(x) it would be.

Suppose you let the terms go on forever?

$$P(x) = a_0 + a_1x + a_2x^2 + \ldots + a_{n-1}x^{n-1} + a_nx^n + \ldots$$

This we call an “Infinite Series”.

And, then, maybe it would be a perfect representation of a function f(x).

In fact, this is the case. And, starting in the 17th century our ancestors used the infinite series representation of functions to do their analysis.

There are serious questions of when this is "legitimate" in the sense that it does not lead to contradictions or nonsense, which it sometimes does if one is not careful, whatever that means.

In the 19th century our ancestors figured this all out and made the use of infinite series rigorous just as they made all of mathematics rigorous. If you are a mathematics major you will learn all about this.

In the meantime, STEM students may just trust that what we tell them will work if they follow our "rules".

The problem, of course, is to find the proper infinite series to represent a function f(x).

It turns out that this is very easy with derivatives.

Newton used a generalization of the Binomial Theorem with non integer exponents to great advantage. But, this is now superseded by something we call Taylor Infinite Series representation of a function, f(x).

And, this then extends into complex numbers, and is indeed how we extend the definition of such functions as sin(x) and cos(x) into the complex number domain. Recall our treatment in Tier 4.

$$e^{ix} = \mathrm{Cos}(x) + i\mathrm{Sin}(x)$$

Let f(x) be a function with derivatives of all orders.

$f^{(n)}(x)$ means the nth order derivative of f(x)

Suppose that f(x) is equal to an infinite series, "centered at a"

$f(x) = a_0+a_1(x-a) + a_2(x-a)^2 + \ldots + a_{n-1}(x-a)^{n-1} + a_n(x-a)^n + \ldots$

What would the a_i equal?

$a_0 = f(a)$ by substituting $x = a$ into both sides.

Now, differentiate both sides

$f'(x) = a_1+ 2a_2(x-a)^1 + \ldots +(n-1)a_{n-1}(x-a)^{n-2} + na_n(x-a)^{n-1} + \ldots$

and $a_1= f'(a) = f^{(1)}(x)$

What do you suppose a_2 will equal?

$f^{(2)}(x) = 2a_2 + 3x2a_3(x-a)^1 \ \ldots + (n-2)(n-1)a_{n-1}(x-a)^{n-3} + (n-1)na_n(x-a)^{n-2} + \ldots$

$2a_2 = f^{(2)}(a)$ or $a_2 = f^{(2)}(a)/2!$

Differentiate both sides again and get

$a_3 = f^{(3)}(a)/3!$

In general, $a_n = f^{(n)}(a)/n!$

So we write $f(x) = \text{SUM } [f^{(n)}(a)/n!](x - a)^n$

Σ is used for SUM and it is understood that n ranges from 0 to ∞

However, when one actually calculates with such a series, one always truncates and only uses as high a power of n as needed for the approximation.

$f(x) = a_0+a_1(x-a) + a_2(x-a)^2 + . . . + a_{n-1}(x-a)^{n-1} + a_n(x-a)^n + O(f^{(n+1)}(a)\ (x-a)^{n+1}/(n+1)!)$

$O(f^{(n+1)}(a)\ (x-a)^{n+1}/(n+1)!)$ is the estimate of the largest error that will happen with this approximation.

In the old days, we worked pretty hard to find the Taylor expansion of various functions, f(x).

When a = 0 we call this a Maclaurin series.

Now WA makes this all very easy.

WA series sin(x)

WA series cos(x)

WA series e^x

WA series sin(x) at x = pi/4

WA series e^ix

WA series sin(x) at x = 0 order 20

WA e^ix

Tier 5 Calculus Lesson 11 Exercises: Series Expansions

Q1. What is the general form for the Taylor series?

Q2. What is understood to be the range for the Taylor series?

Q3. The Taylor series centers around “a”. To turn the Taylor series into the Maclaurin series, what must “a” equal?

Use Wolfram Alpha to find the Taylor series expansion for the following equations. Also find the first listed series representations.

Q4. f(x) = tan(x)

Q5. f(x) = sin(x)/cos(x)

Q6. f(x) = sin(e^x)

Q7. f(x) = tan(x) at x=0 order 12

Q8. f(x) = sin(x)/cos(x) order 10

Q9. f(x) = sin(e^x) order 15

Q10. sin(x)/cos(x) at x = pi/4

A1. f(x) = SUM [$f^{(n)}$(a)/n!](x – a)^n

A2. 0 to ∞

A3. 0

A4.

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7)$$

(Taylor series)

A5.

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^7)$$

(Taylor series)

A6.

$$\sin(1) + x\cos(1) + \frac{1}{2}x^2(\cos(1) - \sin(1)) - \frac{1}{2}x^3\sin(1) + \frac{1}{24}x^4(-6\sin(1) - 5\cos(1)) + \frac{1}{120}x^5(-5\sin(1) - 23\cos(1)) + O(x^6)$$

(Taylor series)

$$\sin(e^x) = \sin\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)$$

A7.

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1382x^{11}}{155925} + \frac{21844x^{13}}{6081075} + O(x^{14})$$

(Taylor series)

A8.

$$x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + \frac{62x^9}{2835} + \frac{1382x^{11}}{155925} + O(x^{12})$$

(Taylor series)

A9.

$$\sin(1) + x\cos(1) + \frac{1}{2}x^2(\cos(1) - \sin(1)) - \frac{1}{2}x^3\sin(1) + \frac{1}{24}x^4(-6\sin(1) - 5\cos(1)) + \frac{1}{120}x^5(-5\sin(1) - 23\cos(1)) + \frac{1}{720}x^6(33\sin(1) - 74\cos(1)) + \frac{1}{720}x^7(38\sin(1) - 23\cos(1)) + \frac{x^8(1309\sin(1) + 57\cos(1))}{40\,320} + \frac{x^9(4905\sin(1) + 3466\cos(1))}{362\,880} + \frac{x^{10}(11\,516\sin(1) + 27\,361\cos(1))}{3\,628\,800} + \frac{x^{11}(4709\cos(1) - 695\sin(1))}{1\,209\,600} + x^{12}\left(\frac{85\,961\cos(1)}{59\,875\,200} - \frac{25\sin(1)}{21\,504}\right) + \frac{x^{13}(1\,888\,525\cos(1) - 4\,932\,512\sin(1))}{6\,227\,020\,800} + \frac{x^{14}(-32\,889\,885\sin(1) - 4\,974\,059\cos(1))}{87\,178\,291\,200} + \frac{x^{15}(-174\,282\,151\sin(1) - 134\,695\,952\cos(1))}{1\,307\,674\,368\,000} + O(x^{16})$$

(Taylor series)

$$\sin(e^x) = \sin\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right)$$

A10.

$$1 + 2\left(x - \frac{\pi}{4}\right) + 2\left(x - \frac{\pi}{4}\right)^2 + \frac{8}{3}\left(x - \frac{\pi}{4}\right)^3 + \frac{10}{3}\left(x - \frac{\pi}{4}\right)^4 + \frac{64}{15}\left(x - \frac{\pi}{4}\right)^5 + O\left(\left(x - \frac{\pi}{4}\right)^6\right)$$

(Taylor series)

Tier 5 Calculus Lesson 12 Notes: Final Thoughts on Derivatives

The first obvious application of derivative is when one wants to know the rate of change of a function, i.e. the slope of the tangent line on each point on the graph of the function.

This comes up a lot in virtually all STEM subjects and, indeed, in any subject where one is performing a quantitative analysis of something.

The function is created as a model for some phenomenon one is studying.

Then, the rate of change is calculated using differential calculus and the derivative.

The second application of derivatives which was vitally important in the old days was in graphing functions.

As we learned, a graph is the quickest way to really understand the behavior of a function, and consequently, the phenomenon the function is modeling.

Historically, graphing functions was very tedious and time consuming and consumed much of the differential calculus training.

For example, just to find the maxima and minima one had to find the zeros of the derivative, which itself could be very difficult, and then apply other tests to determine which stationary (critical) points were maxima or minima or just points of inflection.

Derivatives can be used to determine when the function is increasing and decreasing.

Derivatives also can be used to determine concavity and points of inflection.

!!! Today, tools like Wolfram Alpha obviate the need for these classical manual procedures.

One can now graph functions virtually instantaneously including functions whose graphs were essentially intractable historically.

A significant percentage of the manual graphing techniques taught in a classical calculus course are now effectively obsolete.

No modern employer would pay you to apply these techniques. They were great for our ancestors before the modern tools, but like many of the classical tools, no longer useful.

The third, and more important, application of derivatives is in calculating integrals which we will learn in the second half of this course.

Anti-Derivatives are extremely important in many science and engineering applications involving integrals.

As you will learn, anti-derivatives are what we use in integral calculus thanks to the Fundamental Theorem of Calculus.

A fourth application of derivatives is in the representation of functions as infinite series known as Taylor series.

This in turn is what makes Complex Numbers and Complex Analysis work.

Most importantly, Derivatives are utilized in what are called differential equations which are the workhorses of modern science and engineering models. Tier 6.

Indeed, if you go into any STEM subject or any other subject utilizing quantitative analysis you will utilize differential equations.

Fortunately today, tools like Wolfram Alpha make the calculations with derivatives automatic and very easy so one can focus on their meaning and applications.

Historically, one often had to avoid certain calculus problems simply because we did not have adequate tools to solve them.

Indeed, many of the problems in calculus and science textbooks are rigged so we can solve them with the classical manual tools.

Today, with tools like WA we are no longer restricted. Now we can solve virtually any calculus problem that comes up in any application.   Quickly and easily.

Wolfram Alpha does for calculus what the scientific calculator does for arithmetic and what the spreadsheet does for modern accounting and data analysis.

As a STEM professional you will be finding applications of derivatives the rest of your life. There are an unlimited number of examples throughout all STEM subjects.

It is imperative you master a tool such as Wolfram Alpha.

This will become even more important and significant as you learn Integral Calculus

I estimate Wolfram Alpha eliminates as much as 75% of the required work to master and implement calculus compared to the classical approach.

That is why this entire calculus program can be comfortably completed in about one semester.

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Increasing / Decreasing

Max / Min

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T5 C8 Implicit Differentiation

T5 C9 Relative Rates of Change

T5 C10 Inverse Functions

T5 C11 Series Expansions

T5 C12 Final Thoughts on Derivatives

Tier 5 Calculus Lesson 12 Exercises: Final Thoughts on Derivatives

Q1. What is the purpose of a function?

Q2. What is the first use of derivatives?

Q3. What is the second use of derivatives?

Q4. What parts of a graph do derivatives allow you to discover?

Q5. What does a graph allow us to see/do?

Q6. What program allows us to solve practically any calculus problem we can come up with?

A1. To create a model of the phenomenon you are studying

A2. Calculate the rate of change

A3. Graph functions

A4. Is the function increasing/decreasing, concavity, points of inflection, and maxima/minima

A5. Understand the behavior of a function and the phenomenon the function is modeling

A6. Wolfram Alpha